2.4 METHOD OF ITERATION

The method of iteration can be applied to find a real root of the equation $f(x) = 0$ by rewriting the same in the form,

$$x = \phi(x) \tag{2.3}$$

For example, $f(x) = \cos x - 2x + 3 = 0$. It can be rewritten as

$$x = \frac{1}{2}(\cos x + 3) = \phi(x)$$

Let $x = \xi$ is the desired root of Eq. (2.3). Suppose x_0 is its initial approximation. The first and successive approximations to the root can be obtained as

$$\left.\begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ &\vdots \\ x_{n+1} &= \phi(x_n) \end{aligned}\right\} \tag{2.4}$$

Definition 2.1 Let $\{x_i\}$ be the sequence obtained by a given method and let $x = \xi$ denotes the root of the equation $f(x) = 0$. Then, the method is said to be *convergent*, if and only if

$$\operatorname*{Lt}_{n \to \infty} |x_n - \xi| = 0$$

The convergence of the above sequence to the root is stated as in Theorem 2.1.

Theorem 2.1 Support $x = \xi$ be a root of the equation $f(x) = 0$, which can be rewritten as $x = \phi(x)$, contained in an interval I. Also, let $\phi(x)$ and $\phi'(x)$ be continuous in I. Then, if $|\phi'(x)| < 1$ for all x in I, the iterative process defined by $x_{n+1} = \phi(x_n)$ converges to the root $x = \xi$, if and only if, the initially chosen approximation $x_0 \in I$.

This method is illustrated through the following examples.

Example 2.5 Use the method of iteration to determine the real root of the equation $e^{-x} = 10x$ correct to four decimal places.

Solution Let $f(x) = e^{-x} - 10x = 0$, we observe that $f(0) = 1$ and $f(1) = -9.6321$. Since $f(0) < f(1)$ numerically, the root is near to $x = 0$. Now, we shall rewrite the given equation in the form

$$x = \frac{1}{10}e^{-x} = \phi(x)$$

Therefore,

$$\phi'(x) = -\frac{1}{10}e^{-x}$$

and

$$|\phi'(x)| = \frac{1}{10}e^{-x} = \frac{1}{10\,e^{x}} < 1$$

for all x in (0, 1). Hence, the method of iteration can be applied. Thus, we start with the initial value $x_0 = 0$, then

$$x_1 = \phi(x_0) = \frac{1}{10} = 0.1, \quad f(x_1) = -0.09516$$

Similarly, the successive approximations are

$$x_2 = \phi(x_1) = \frac{1}{10}e^{-0.1} = \frac{0.904837}{10} = 0.09048, \; f(x_2) = 0.00869$$

$$x_3 = \phi(x_2) = 0.091349, \quad f(x_3) = -7.90877 \times 10^{-4}$$

$$x_4 = \phi(x_3) = 0.091274, \quad f(x_4) = 2.75784 \times 10^{-5}.$$

Hence, the required root is 0.0913.

Example 2.6 Find a real root of the equation

$$f(x) = x^3 + x^2 - 1 = 0$$

by the method of iteration.

Solution We observe that $f(0) = -1$, $f(1) = 1$ which shows that there is a real root between $x = 0$ and $x = 1$. To find the real root, we rewrite the equation in the form

$$x^2(x + 1) = 1 \quad \text{or} \quad x = \frac{1}{\sqrt{x+1}} = \phi(x)$$

Therefore,

$$\phi'(x) = -\frac{1}{2(x+1)^{3/2}}$$

We note that $|\phi'(x)| < 1$, for all x in (0, 1). Hence, the method of iteration is applicable here.

Taking the initial value $x_0 = 1$, we successively obtain the following values:

$$\begin{aligned}
x_1 &= \phi(x_0) = 1/\sqrt{2} = 0.70711, & f(x_1) &= -0.14644\\
x_2 &= \phi(x_1) = 0.76537, & f(x_2) &= 0.03414\\
x_3 &= \phi(x_2) = 0.75263, & f(x_3) &= 7.2213 \times 10^{-3}\\
x_4 &= \phi(x_3) = 0.75536, & f(x_4) &= 1.55658 \times 10^{-3}\\
x_5 &= \phi(x_4) = 0.75477, & f(x_5) &= -3.44323 \times 10^{-4}\\
x_6 &= \phi(x_5) = 0.7549, & f(x_6) &= 7.38295 \times 10^{-5}
\end{aligned}$$

Hence, the required root is 0.7549.

Note: The given equation can be rewritten in many ways. Suppose, we rewrite

$$x^2 = 1 - x^3 \quad \text{or} \quad x = (1 - x^3)^{1/2} = \phi(x)$$

Then

$$|\phi'(x)| = \frac{3x^2}{2(1 - x^3)^{1/2}}$$

if we take $x = 1$, in the interval (0, 1), $|\phi'(x)| = \infty$, then the condition $|\phi'(x)| < 1$ is violated.

2.5 NEWTON–RAPHSON METHOD

This is a very powerful method for finding the real root of an equation in the form, $f(x) = 0$. Suppose, x_0 is an approximate root of $f(x) = 0$. Let $x_1 = x_0 + h$, where h is small, be the exact root of $f(x) = 0$, then $f(x_1) = 0$. Now, expanding $f(x_0 + h)$ by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \cdots = 0 \tag{2.5}$$

Since h is small, we neglect terms containing h^2 and its higher powers, then

$$f(x_0) + h f'(x_0) = 0 \quad \text{or} \quad h = \frac{-f(x_0)}{f'(x_0)}$$

Therefore, a better approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Still better and successive approximations $x_2, x_3, \ldots, x_n$ to the root can obviously be obtained from the iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.6}$$

This is known as Newton–Raphson iteration formula, which has the following geometrical interpretation:

Suppose, the graph of the function $y = f(x)$ crosses the x-axis at α (see Fig. 2.4), then $x = \alpha$ is the root of the equation $f(x) = 0$.

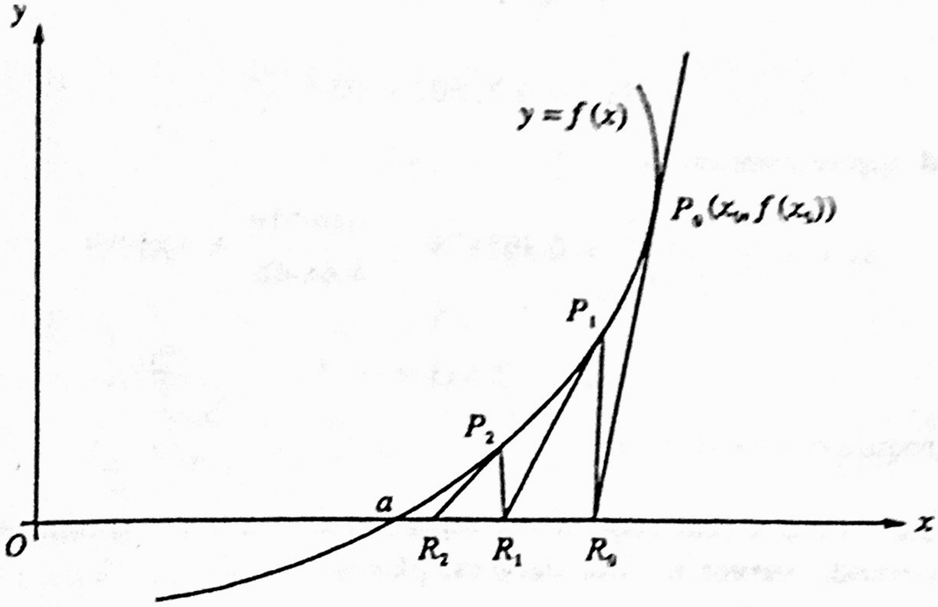


Fig. 2.4 Geometrical interpretation of Newton–Raphson method.

Let x_0 be a point closer to the root α, then the equation of the tangent at $P_0(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)\,(x - x_0) \tag{2.7}$$

This tangent cuts the x-axis at $R_0\ (x_1, 0)$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.8}$$

which is a first approximation to the root α. If P_1 is a point on the curve corresponding to x_1, then the tangent at P_1 cuts the x-axis at $R_1(x_2, 0)$, which is still closer to α, than x_1. Therefore, x_2 is a second approximation to the root. Continuing this process, we arrive at the root α, very rapidly, which is evident from Fig. 2.4. Thus, in this method, we have replaced the part of the curve between the point P_0 and x-axis by a tangent to the curve at P_0 and so on. In order to illustrate this method, we shall consider the following examples.

Example 2.7 Find the real root of the equation $xe^x - 2 = 0$ correct to two decimal places, using Newton–Raphson method.

Solution Given $f(x) = xe^x - 2$, we have

$$f'(x) = xe^x + e^x \text{ and } f''(x) = xe^x + 2e^x$$

clearly, we have

$$f(0) = -2 \quad \text{and} \quad f(1) = e - 2 = 0.71828$$

Hence, the required root lies in the interval (0, 1) and is nearer to 1.

Also, $f(x)$ and $f'(x)$ do not vanish in (0, 1) and $f(x)$ and $f'(x)$ will have the same sign at $x = 1$. Therefore, we take the first approximation $x_0 = 1$, and using Newton–Raphson method, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{e+2}{2e} = 0.867879$$

and

$$f(x_1) = 6.71607 \times 10^{-2}$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.867879 - \frac{0.06716}{4.44902} = 0.85278$$

and

$$f(x_2) = 7.655 \times 10^{-4}$$

Thus, the required root is 0.853.

Example 2.8 Find a real root of the equation $x^3 - x - 1 = 0$ using Newton–Raphson method, correct to four decimal places.

Solution Let $f(x) = x^3 - x - 1$, then we observe that $f(1) = -1, f(2) = 5$. Therefore, the root lies in the interval (1, 2). We also observe

$$f'(x) = 3x^2 - 1, \quad f''(x) = 6x$$

and

$$f(1) = -1, \quad f''(1) = 6, \quad f(2) = 5, \quad f''(2) = 12$$

Since $f(2)$ and $f''(2)$ are of the same sign, we choose $x_0 = 2$ as the first approximation to the root. The second approximation is computed using Newton–Raphson method as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{5}{11} = 1.54545 \quad \text{and} \quad f(x_1) = 1.14573$$

The successive approximations are

$$x_2 = 1.54545 - \frac{1.14573}{6.16525} = 1.35961, \quad f(x_2) = 0.15369$$

$$x_3 = 1.35961 - \frac{0.15369}{4.54562} = 1.32579, \quad f(x_3) = 4.60959 \times 10^{-3}$$

$$x_4 = 1.32579 - \frac{4.60959 \times 10^{-3}}{4.27316} = 1.32471, \quad f(x_4) = -3.39345 \times 10^{-5}$$

$$x_5 = 1.32471 + \frac{3.39345 \times 10^{-5}}{4.26457} = 1.324718, \quad f(x_5) = 1.823 \times 10^{-7}$$

Hence, the required root is 1.3247.

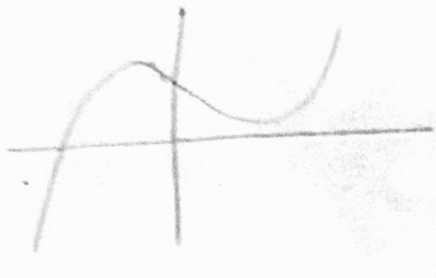
Convergence of Newton–Raphson method

To examine the convergence of Newton–Raphson formula (2.6), that is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We compare it with the general iteration formula $x_{n+1} = \phi(x_n)$, and thus obtain

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$



In general, we write it as

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

We have already noted in Theorem 2.1 that the iteration method converges if $|\phi'(x)| < 1$. Therefore, Newton–Raphson formula (2.6) converges, provided

$$|f(x)f''(x)| < |f'(x)|^2 \qquad (2.9)$$

in the interval considered. Newton–Raphson formula therefore converges, provided the initial approximation x_0 is chosen sufficiently close to the root and $f(x), f'(x)$ and $f''(x)$ are continuous and bounded in any small interval containing the root.

Definition 2.2 Let

$$x_n = \alpha + \varepsilon_n, \qquad x_{n+1} = \alpha + \varepsilon_{n+1}$$

where α is a root of $f(x) = 0$. If we can prove that $\varepsilon_{n+1} = K\varepsilon_n^p$, where K is a constant and ε_n is the error involved at the nth step, while finding the root by an iterative method, then the rate of convergence of the method is p.

We can now establish that Newton–Raphson method converges quadratically. Let

$$x_n = \alpha + \varepsilon_n, \qquad x_{n+1} = \alpha + \varepsilon_{n+1}$$

where α is a root of $f(x) = 0$ and ε_n is the error involved at the nth step, while finding the root by Newton–Raphson formula (2.6). Then, Eq. (2.6) gives,

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

i.e.,

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \frac{\varepsilon_n f'(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

Using Taylor's expansion, we get

$$\varepsilon_{n+1} = \frac{1}{f'(\alpha) + \varepsilon_n f''(\alpha) + \cdots} \left\{ \varepsilon_n [f'(\alpha) + \varepsilon_n f''(\alpha) + \cdots] - \left[f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2} f''(\alpha) + \cdots \right] \right\}$$

Since α is a root, $f(\alpha) = 0$. Therefore, the above expression simplifies to

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2} f''(\alpha) \frac{1}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[1 + \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$\approx \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right]$$

or

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} + O\left(\varepsilon_n^3\right)$$

On neglecting terms of order ε_n^3 and higher powers, we obtain

$$\varepsilon_{n+1} = K\varepsilon_n^2 \tag{2.10}$$

where

$$K = \frac{f''(\alpha)}{2f'(\alpha)} \tag{2.11}$$

It shows that Newton–Raphson method has second order convergence or converges quadratically.

Example 2.9 Set up Newton's scheme of iteration for finding the square root of a positive number N.

Solution The square root of N can be carried out as a root of the equation $x^2 - N = 0$. Let $f(x) = x^2 - N$. By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In this problem, $f(x) = x^2 - N$, $f'(x) = 2x$. Therefore,

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \tag{2.12}$$

Example 2.10 Evaluate $\sqrt{12}$, by Newton's formula.

Solution Since $\sqrt{9} = 3$, $\sqrt{16} = 4$, we take $x_0 = (3 + 4)/2 = 3.5$. Using Eq. (2.12), we have

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643$$

$$x_2 = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Hence, $\sqrt{12} = 3.4641$.

Example 2.11 Obtain the Newton–Raphson extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$$

for finding the root of the equation $f(x) = 0$.

Solution Expanding $f(x)$ by Taylor's series, in the neighbourhood of x_0, we obtain after retaining the first order term only

$$0 = f(x) = f(x_0) + (x - x_0) f'(x_0) + \cdots$$

Which gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This is the first approximation to the root. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.13}$$

Again, expanding $f(x)$ by Taylor's series and retaining up to second order term, we have

$$0 = f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0)$$

Therefore,

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2} f''(x_0) = 0$$

Using Eq. (2.13), the above equation reduces to the form

$$f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0) = 0$$

Thus, the Newton–Raphson extended formula is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0) \tag{2.14}$$

This is also known as Chebyshev's formula of third order.