2.3 REGULA–FALSI METHOD

This method is also known as the *method of false position.* In this method, we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs. Intermediate value property suggests that the graph of $y = f(x)$ crosses the x-axis between these two points, and therefore, a root say $x = \xi$ lies between these two points. Thus, to find a real root of $f(x) = 0$ using Regula-Falsi method, we replace the part of the curve between the points $A[x_n, f(x_n)]$ and $B[x_{n-1}, f(x_{n-1})]$ by a chord in that interval and we take the point of intersection of this chord with the x-axis as a first approximation to the root (see Fig. 2.3).

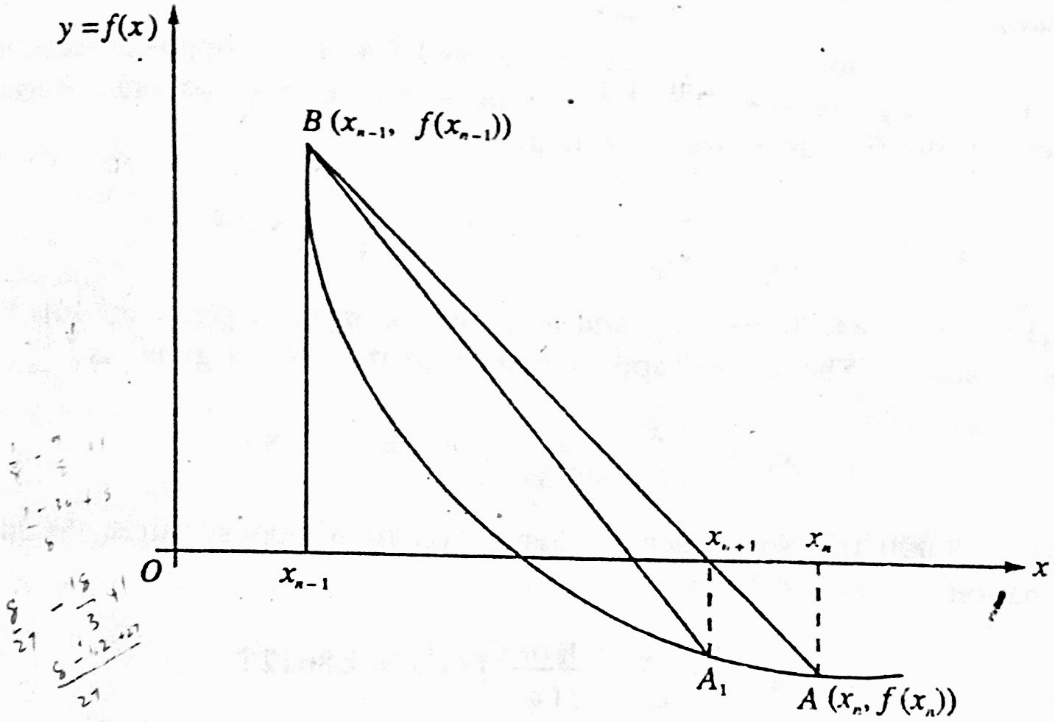

Fig. 2.3 Geometrical illustration of Regula-Falsi method.

Now, the equation of the chord joining the points A and B is

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n} \tag{2.1}$$

Setting $y = 0$ in Eq. (2.1), we get

$$x = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n)$$

Hence, the first approximation to the root of $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \tag{2.2}$$

From Fig. 2.3, we observe that $f(x_{n-1})$ and $f(x_{n+1})$ are of opposite sign. Thus, it is possible to apply the above procedure, to determine the line through B and A_1 and so on. Hence, the successive approximations to the root of $f(x) = 0$ is given by Eq. (2.2). This method can best be understood through the following examples.

Example 2.2 Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$,

(i) if the root lies between 2 and 4
(ii) if the root lies between 2 and 3.

Comment on the results.

Solution Let $f(x) = x^3 - 9x + 1$.

(i) $f(2) = -9$ and $f(4) = 29$. Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4. Taking $x_1 = 2$, $x_2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{2 \times 29}{38} = 2.47368$$

and $f(x_3) = -6.12644$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.73989$$

and $f(x_4) = -3.090707$. Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.86125$$

and $f(x_5) = -1.32686$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.47368	−6.12644
3	2.73989	−3.090707
4	2.86125	−1.32686

(ii) $f(2) = -9$ and $f(3) = 1$. Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3. Taking $x_1 = 2$, $x_2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{1}{10} = 2.9$$

and $f(x_3) = -0.711$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.94156$$

and $f(x_4) = -0.0207$. Now, we observe that $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.94275$$

and $f(x_5) = -0.0011896$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.9	−0.711
3	2.94156	−0.0207
4	2.94275	−0.0011896

From the above computations, we observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_5, when the interval considered is (2, 3), is closer to the root. Hence, an important observation in this method is that the interval (x_1, x_2) chosen initially in which the root of the equation lies must be sufficiently small.

Example 2.3 Use Regula-Falsi method to find a real root of the equation

$$\log x - \cos x = 0$$

accurate to four decimal places after three successive approximations.

Solution Given $f(x) = \log x - \cos x$. We observe that

$$f(1) = 0 - 0.5403 = -0.5403$$

and

$$f(2) = 0.69315 + 0.41615 = 1.1093$$

Since $f(1)$ and $f(2)$ are of opposite signs, the root lies between $x_1 = 1$, $x_2 = 2$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2 - \frac{1.1093}{1.6496} = 1.3275$$

and

$$f(x_3) = 0.2833 - 0.2409 = 0.0424$$

Now, since $f(x_1)$ and $f(x_3)$ are of opposite signs, the second approximation is obtained as

$$x_4 = 1.3275 - \frac{(.3275)\,(.0424)}{0.0424 + 0.5403} = 1.3037$$

and

$$f(x_4) = 1.24816 \times 10^{-3}$$

Similarly, we observe that $f(x_1)$ and $f(x_4)$ are of opposite signs, so, the third approximation is given by

$$x_5 = 1.3037 - \frac{(0.3037)\,(0.001248)}{0.001248 + 0.5403} = 1.3030$$

and

$$f(x_5) = 0.62045 \times 10^{-4}$$

Hence, the required real root is 1.3030.

Example 2.4 Using Regula-Falsi method, find the real root of the following equation correct to three decimal places:

$$x \log_{10} x = 1.2$$

Solution Let $f(x) = x \log_{10} x - 1.2$. We observe that $f(2) = -0.5979$, $f(3) = 0.2314$. Since $f(2)$ and $f(3)$ are of opposite signs, the real root lies between $x_1 = 2$, $x_2 = 3$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{0.2314}{0.8293} = 2.72097$$

and $f(x_3) = -0.01713$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root of $f(x) = 0$ lies between x_2 and x_3. Now, the second approximation is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.7402$$

and $f(x_4) = -3.8905 \times 10^{-4}$. Thus, the root of the given equation correct to three decimal places is 2.740.