Chapter 2

Solution of Algebraic and Transcendental Equations

2.1 INTRODUCTION

One of the basic problems in science and engineering is the computation of roots of an equation in the form, $f(x) = 0$. The equation $f(x) = 0$ is called an *algebraic equation*, if it is purely a polynomial in x; it is called a *transcendental equation* if $f(x)$ contains trigonometric, exponential or logarithmic functions. For example,

$$x^3 + 5x^2 - 6x + 3 = 0$$

is an algebraic equation, whereas

$$M = E - e \sin E \quad \text{and} \quad ax^2 + \log (x - 3) + e^x \sin x = 0$$

are transcendental equations.

To find the solution of an equation $f(x) = 0$, we find those values of x for which $f(x) = 0$ is satisfied. Such values of x are called the *roots* of $f(x) = 0$. Thus a is a root of an equation $f(x) = 0$, if and only if, $f(a) = 0$.

Before, we develop various numerical methods, we shall list below some of the basic properties of an algebraic equation:

(i) Every algebraic equation of nth degree, where n is a positive integer, has n and only n roots.

(ii) Complex roots occur in pairs. That is, if $(a + ib)$ is a root of $f(x) = 0$, then $(a - ib)$ is also a root of this equation.

(iii) If $x = a$ is a root of $f(x) = 0$, a polynomial of degree n, then $(x - a)$ is a factor of $f(x)$. On dividing $f(x)$ by $(x - a)$ we obtain a polynomial of degree $(n - 1)$.

(iv) Descartes rule of signs: The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficients cannot exceed the number of changes in sign of the coefficients in the polynomial $f(x) = 0$. Similarly, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes in the sign of the coefficients of $f(-x) = 0$. For example, consider an equation

$$x^3 - 3x^2 + 4x - 5 = 0$$

As there are three changes in sign, also, the degree of the equation three, and hence the given equation will have all the three positi roots.

(v) Intermediate value property: If $f(x)$ is a real valued continuo function in the closed interval $a \leq x \leq b$. If $f(a)$ and $f(b)$ have oppos signs, then the graph of the function $y = f(x)$ crosses the x-axis at le once; that is $f(x) = 0$ has at least one root ξ such that $a < \xi < b$.

Broadly speaking, all the known numerical methods for solving either transcendental equation or an algebraic equation can be classified into tw groups: *direct methods* and *iterative methods*. Direct methods require r knowledge of the initial approximation of a root of the equation $f(x) = 0$, whi iterative methods do require first approximation to initiate iteration. How to g the first approximation? We can find the approximate value of the root of $f(x) =$ either by a *graphical method* or by an *analytical method* as explained belo

Graphical method

Often, the equation $f(x) = 0$ can be rewritten as $f_1(x) = f_2(x)$ and the fir approximation to a root of $f(x) = 0$ can be taken as the abscissa of the point o intersection of the graphs of $y = f_1(x)$ and $y = f_2(x)$. For example, consider,

$$f(x) = x - \sin x - 1 = 0$$

It can be written as $x - 1 = \sin x$. Now, we shall draw the graphs of

$$y = x - 1 \qquad \text{and} \qquad y = \sin x$$

as shown in Fig. 2.1. The approximate value of the root is found to be 1.9.

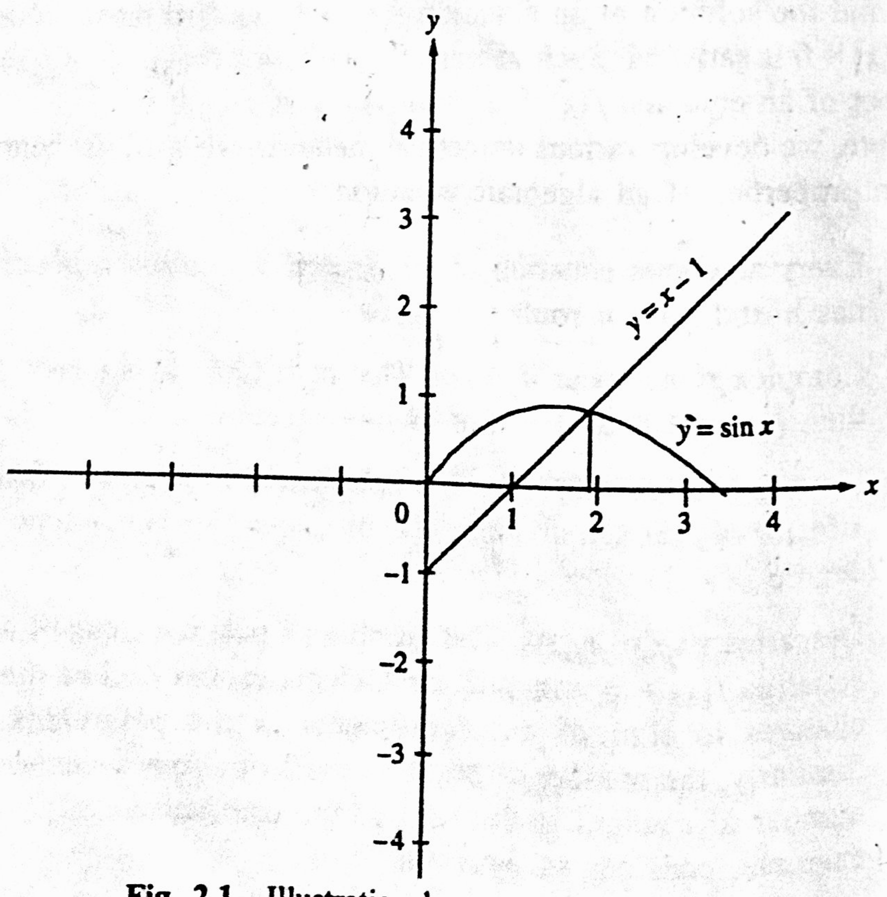

Fig. 2.1 Illustration by graphical method.

Analytical method

This method is based on 'intermediate value property'. We shall illustrate it through an example. Let,

$$f(x) = 3x - \sqrt{1 + \sin x} = 0$$

We can easily verify

$$f(0) = -1$$

$$f(1) = 3 - \sqrt{1 + \sin\left(1 \times \frac{180}{\pi}\right)} = 3 - \sqrt{1 + 0.84147} = 1.64299$$

We observe that $f(0)$ and $f(1)$ are of opposite signs. Therefore, using intermediate value property we infer that there is at least one root between $x = 0$ and $x = 1$. This method is often used to find the first approximation to a root of either transcendental equation or algebraic equation. Hence, in analytical method, we must always start with an initial interval (a, b), so that $f(a)$ and $f(b)$ have opposite signs.

2.2 BISECTION METHOD

This method is due to Bolzano. Suppose, we wish to locate the root of an equation $f(x) = 0$ in an interval, say (x_0, x_1). Let $f(x_0)$ and $f(x_1)$ are of opposite signs, such that $f(x_0)\, f(x_1) < 0$.

Then the graph of the function crosses the x-axis between x_0 and x_1, which guarantees the existence of at least one root in the interval (x_0, x_1). The desired root is approximately defined by the mid-point

$$x_2 = \frac{x_0 + x_1}{2}$$

If $f(x_2) = 0$, then x_2 is the desired root of $f(x) = 0$. However, if $f(x_2) \neq 0$, then the root may be between x_0 and x_2 or x_2 and x_1. Now, we define the next approximation by

$$x_3 = \frac{x_0 + x_2}{2}$$

provided $f(x_0)\, f(x_2) < 0$, then the root may be found between x_0 and x_2 or by

$$x_3 = \frac{x_1 + x_2}{2}$$

provided $f(x_1)\, f(x_2) < 0$, then the root lies between x_1 and x_2 etc.

Thus, at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in Fig. 2.2. This process of halving the intervals is continued to determine a smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root. This method is illustrated in the following example.

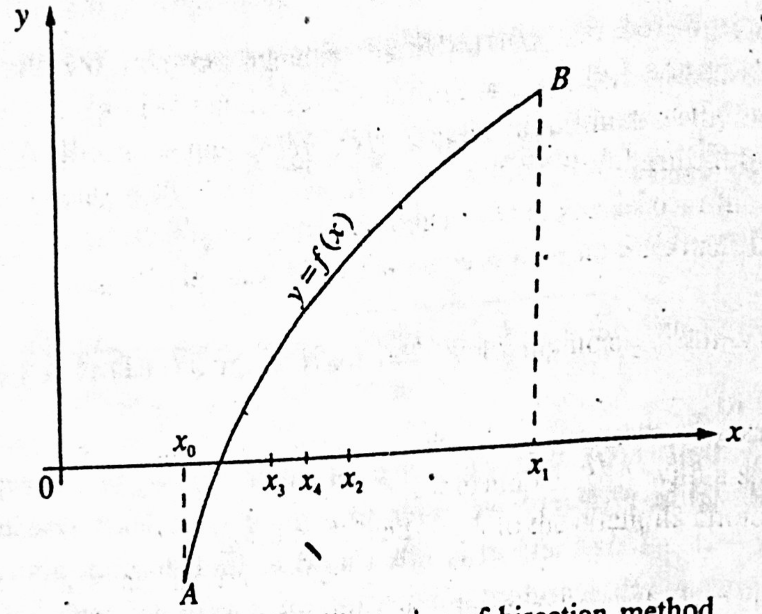

Fig. 2.2 Geometrical illustration of bisection method.

Example 2.1 Solve $x^3 - 9x + 1 = 0$ for the root between $x = 2$ and $x = 4$ by the bisection method.

Solution Given $f(x) = x^3 - 9x + 1$. We can verify $f(2) = -9$, $f(4) = 29$. Therefore, $f(2)\,f(4) < 0$ and hence the root lies between 2 and 4. Let $x_0 = 2$, $x_1 = 4$. Now, we define

$$x_2 = \frac{x_0 + x_1}{2} = \frac{2 + 4}{2} = 3$$

as a first approximation to a root of $f(x) = 0$ and note that $f(3) = 1$, so that $f(2)\,f(3) < 0$. Thus, the root lies between 2 and 3. We further define,

$$x_3 = \frac{x_0 + x_2}{2} = \frac{2 + 3}{2} = 2.5$$

and note that $f(x_3) = f(2.5) < 0$, so that $f(2.5)\,f(3) < 0$. Therefore, we define the mid-point,

$$x_4 = \frac{x_3 + x_2}{2} = \frac{2.5 + 3}{2} = 2.75, \text{ etc.}$$

Similarly, we find that

$$x_5 = 2.875 \quad \text{and} \quad x_6 = 2.9375$$

and the process can be continued until the root is obtained to the desired accuracy. These results are presented in the table.

n	x_n	$f(x_n)$
2	3	1.0
3	2.5	–5.875
4	2.75	–2.9531
5	2.875	–1.1113
6	2.9375	–0.0901