2.9 SYSTEM OF NON-LINEAR EQUATIONS

The general problem is to solve a system of n non-linear equations in unknowns. That is, to solve

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}\right\} \tag{2.46}$$

Most of the known methods are of iterative type, where we start with initial guess and improve the solution iteratively, until a required tolerance is achieved. By taking the gradients of all the variables, we get a function matrix called the Jacobian defined as

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

It is essential that this Jacobian be non-singular for the existence of a solution. It may also be noted that the value of the Jacobian changes from iteration to iteration. Suppose, we denote the vector of components $(x_1, x_2, \ldots, x_n)$ by X and the vector $(f_1, f_2, \ldots, f_n)$ by F, then the system (2.46) can be written as

$$F(X) = 0 \tag{2.47}$$

The task is to solve the system (2.47). We present below a simple Newton's method by considering a system in two variables such as

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned} \tag{2.48}$$

Suppose, we choose (x_0, y_0) as the initial (guess) approximation and let h, k are the quantities to be determined such that

$$\begin{aligned} f(x_0 + h,\ y_0 + k) &= 0 \\ g(x_0 + h,\ y_0 + k) &= 0 \end{aligned} \tag{2.49}$$

Using Taylor series expansion, we get

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\Bigg|_{(x_0, y_0)} + \text{H.O.T} = 0$$

$$g(x_0 + h, y_0 + k) = g(x_0, y_0) + \left(h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y}\right)\Bigg|_{(x_0, y_0)} + \text{H.O.T} = 0$$

Neglecting the higher order terms (H.O.T) and solving the above system, we get the approximate values of h and k as

$$h = \left. \frac{-f\dfrac{\partial g}{\partial y} + g\dfrac{\partial f}{\partial y}}{J} \right|_{(x_0, y_0)} \tag{2.50}$$

$$k = \left. \frac{-g\dfrac{\partial f}{\partial x} + f\dfrac{\partial g}{\partial x}}{J} \right|_{(x_0, y_0)}$$

where J is the Jacobian defined by

$$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \tag{2.51}$$

Once h and k are computed, we get the first approximate solution in the form

$$\left. \begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 + k \end{aligned} \right\} \tag{2.52}$$

which suggests an iteration

$$\left. \begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + k \end{aligned} \right\} \tag{2.53}$$

Observe that we need to guess a good initial approximation (x_0, y_0) for convergence of iteration, which we may get graphically. The generalisation to a system of equations in n variables is of course straight forward. Here, follows an example for illustration.

Example 2.15 Solve the following system of equations

$$f(x, y) = x^3 - 3xy^2 - 2x + 2 = 0$$

$$g(x, y) = 3x^2y - y^3 - 2y = 0$$

taking $x_0 = y_0 = 1$, as the initial approximation.

Solution In the present example, we have

$$\begin{aligned} f(x, y) &= x^3 - 3xy^2 - 2x + 2 \\ g(x, y) &= 3x^2y - y^3 - 2y \end{aligned} \tag{1}$$

Therefore,

$$\begin{aligned} f_x &= 3x^2 - 3y^2 - 2, \qquad f_y = -6xy \\ g_x &= 6xy, \qquad g_y = 3x^2 - 3y^2 - 2 \end{aligned} \tag{2}$$

Assume the initial approximation $(x_0, y_0) = (1, 1)$ so that

$$f(x_0, y_0) = -2, \qquad g(x_0, y_0) = 0$$

$$f_x\Big|_{(x_0, y_0)} = -2, \quad f_y\Big|_{(x_0, y_0)} = -6$$

$$g_x\Big|_{(x_0, y_0)} = 6, \quad g_y\Big|_{(x_0, y_0)} = -2$$

then the Jacobian

$$J\Big|_{(x_0, y_0)} = \begin{vmatrix} -2 & -6 \\ 6 & -2 \end{vmatrix} = 40 \neq 0$$

Hence, the solution exists. Now, from Eq. (2.50) we compute

$$h = \frac{[(2)(-2) + (0)(-6)}{40} = -\frac{1}{10} = -0.1$$

$$k = \frac{[(0)(-2) + (-2)(6)}{40} = -\frac{3}{10} = -0.3$$

Thus, the first approximation is

$$x_1 = x_0 + h = 1 - 0.1 = 0.9, \; y_1 = y_0 + k = 1 - 0.3 = 0.7$$

Similarly, we find

$$f(x_1, y_1) = 0.729 - 1.323 - 1.8 + 2 = -0.394$$

$$g(x_1, y_1) = 1.701 - 0.343 - 1.4 = -0.042$$

$$f_x = 2.43 - 1.47 - 2 = -1.04, \quad f_y = -3.78$$

$$g_x = 3.78, \; g_y = 2.43 - 1.47 - 2 = -1.04$$

Therefore,

$$J = \begin{vmatrix} -1.04 & -3.78 \\ 3.78 & -1.04 \end{vmatrix} = 15.37 \neq 0$$

$$h = \frac{(0.394)(-1.04) + (-0.042)(-3.78)}{15.37} = -0.0163$$

$$k = \frac{(0.042)(-1.04) + (-0.394)(3.78)}{15.37} = -0.0997$$

Hence, the second approximation is

$$x_2 = x_1 + h = 0.9 - 0.0163 = 0.8837$$

$$y_2 = y_1 + k = 0.7 - 0.0997 = 0.6003$$

This is the solution of the given system after two iterations. At this point, we may note that

$$f(x_2, y_2) = -0.03265$$

and

$$g(x_2, y_2) = -0.01055.$$

Further continuation gives

$$f_x\big|_{(x_2, y_2)} = 1.2617, \quad f_y\big|_{(x_2, y_2)} = -3.1829$$

$$g_x\big|_{(x_2, y_2)} = 3.1829, \quad g_y\big|_{(x_2, y_2)} = 1.2617$$

and

$$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = 11.7227 \neq 0$$

which yield

$$h = 0.006379, \qquad k = -0.0773$$

Thus, the third approximation is

$$x_3 = x_2 + h = 0.8901, \qquad y_3 = y_2 + k = 0.5926$$

and

$$f(x_3, y_3) = -0.0127, \quad g(x_3, y_3) = 0.0152$$

The three iterations are tabulated as

i	x_i	y_i	$f(x_i, y_i)$	$g(x_i, y_i)$
1	0.9	0.7	–0.394	–0.042
2	0.8837	0.6003	–0.0327	–0.0106
3	0.8901	0.5926	–0.0127	0.0152