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Euler-Lagrange equation for two independent variables

Let $F = F(x, y, u, u_x, u_y)$

where x and y are independent variables.

In this case the problem is to determine the surface which extremizes the functional

$$I[u(x,y)] = \iint_R F(x, y, u, u_x, u_y) dx dy$$

Theorem The extremal (surface) of the functional

$$I[u(x,y)] = \iint_R F(x, y, u, u_x, u_y) dx dy$$

where u is prescribed on the boundary of the region R . Moreover u is supposed to possess continuous partial derivatives up to second order, is given by the PDE

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0.$$

Proof

Let

$$I[u(x,y)] = \iint_R F(x, y, u, u_x, u_y) dx dy \rightarrow ①$$

(2)

Then

$$\delta I = \iint_R \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_x} \delta u_x + \frac{\partial F}{\partial u_y} \delta u_y \right) dx dy \rightarrow (2)$$

Consider

$$\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) = \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u + \frac{\partial F}{\partial u_x} \delta u_x$$

$\therefore \frac{\partial F}{\partial u_x} \delta u_x = \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u$

Similarly

$$\frac{\partial F}{\partial u_y} \delta u_y = \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u$$

Therefore (2) becomes

$$\begin{aligned} \delta I &= \iint_R \left[\frac{\partial F}{\partial u} \delta u + \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u \right. \\ &\quad \left. + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] dx dy \end{aligned}$$

On rearranging, we get

$$\begin{aligned} \delta I &= \iint_R \left[\left[\frac{\partial F}{\partial u} \delta u - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] \right] dx dy \\ &\quad + \iint_R \left[\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) \right] dx dy \\ &= I_1 + I_2 \end{aligned}$$

(3)

According to Green's Theorem, we get

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_C (P dx + Q dy)$$

Using Green's theorem

$$\begin{aligned} I_2 &= \iint_R \left[\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) \right] dx dy \\ &= \iint_R \left[\frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial y} \left(-\frac{\partial F}{\partial u_y} \delta u \right) \right] dx dy \\ &= \int_C \left(-\frac{\partial F}{\partial u_y} \delta u dx + \frac{\partial F}{\partial u_x} \delta u dy \right) \\ &= \int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) \delta u \end{aligned}$$

Since u is prescribed on the boundary

Therefore $\delta u = 0$

Thus $I_2 = 0$.

$$\text{Now } \delta I_1 = \iint_R \left[\frac{\partial F}{\partial u} \delta u - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] dx dy$$

For extremal $\delta I = 0$

$$\Rightarrow \iint_R \left[\frac{\partial F}{\partial u} \delta u - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_x} \right) \delta u - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] dx dy = 0$$

(4)

$$\text{or } \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_n} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial y_n} \right) \right] S_{11} du dy = 0$$

Since S_{11} is arbitrary and $S_{11}=0$ at the boundary. Therefore by fundamental theorem of calculus of variations,

$$\boxed{\frac{\partial F}{\partial u} S_{11} - \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u_n} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial y_n} \right) = 0}$$

Plateau's problem

The problem of finding minimal area bounded by a given closed curve is called minimal surface or Plateau's problem.

Let $z = z(u, v)$ be the surface which is described by parameters (u, v) .

An arc element of any curve on it is given by

$$(ds)^2 = E (du)^2 + 2F du dv + G (dv)^2$$

where E, F, G are fundamental quantities of the surface given by

$$E = \frac{\partial \vec{x}}{\partial u} \cdot \frac{\partial \vec{x}}{\partial u}, \quad F = \frac{\partial \vec{x}}{\partial u} \cdot \frac{\partial \vec{x}}{\partial v}, \quad G = \frac{\partial \vec{x}}{\partial v} \cdot \frac{\partial \vec{x}}{\partial v}$$

(5)

If $u=x, v=y$, then

$$E = \left(\frac{\partial \mathbf{r}}{\partial u} \right)^2 = \left(\frac{\partial x}{\partial u} \hat{i} + \frac{\partial z}{\partial u} \hat{k} \right)^2 = 1 + z_x^2$$

$$F = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = (\hat{i} + z_x \hat{k}) \cdot (\hat{j} + z_y \hat{k}) = z_x z_y$$

and

$$G = \left(\frac{\partial \mathbf{r}}{\partial v} \right)^2 = (\hat{j} + z_y \hat{k})^2 = 1 + z_y^2$$

In terms of parametric coordinates (u, v) ,

$$ds = ds_1 ds_2 = \sqrt{E} du \sqrt{G} dv \sin \theta = \sqrt{EG} du dv \sin \theta$$

where θ is angle between parametric coordinate curves $u = \text{constant}$ and $v = \text{constant}$.

$$\text{and } \cos \theta = F/\sqrt{EG}$$

$$\text{Therefore } \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{F^2}{EG}} = \sqrt{\frac{EG - F^2}{EG}}$$

$$\text{and } ds = \sqrt{EG} \sqrt{\frac{EG - F^2}{EG}} du dv$$

$$ds = \sqrt{EG - F^2}$$

when x, y are used as parameters

$$ds = \sqrt{(1 + z_x^2)(1 + z_y^2) - z_x^2 z_y^2} = \sqrt{1 + z_x^2 + z_y^2} dx dy$$

(6)

The problem is to minimize the integral

$$\iint_S (1+z_x^2+z_y^2) dx dy$$

subject to the condition $z = z_0$ on C

On simplifying E-L equation, we get

$$z_{xx} (1+z_y^2) - 2z_x z_y z_{xy} + z_{yy} (1+z_x^2) = 0.$$

9.6 Constrained Extrema

These problems are also called *variational problems with constraints* or *variational problems with side conditions* or *isoperimetric problems*.

There are certain variational problems in which we have to find stationary values of a functional of the form

$$I[y_1(x), y_2(x), \dots, y_n(x)] = \int_{x_1}^{x_2} F(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx$$

where the arguments of F are subject to some additional conditions or constraints such as

(i) $G(x, y_1, \dots, y_n) = \text{constant.}$

or (ii) $G(x, y_1, \dots, y_n, y'_1, \dots, y'_n) = \text{constant.}$

or (iii) $\int_{x_1}^{x_2} G(x, y_1, \dots, y_n, y'_1, \dots, y'_n) dx = \text{constant.}$

Isoperimetric problems are special cases of these problems.

9.6.1 Constrained maxima and minima problems in calculus

Let $u = f(x_1, x_2, \dots, x_n)$ with side conditions

$$\phi_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, m, \quad (m < n)$$

In Lagrange's method of multipliers, we consider an auxiliary function

$$w(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^m \lambda_i \phi_i(x_1, x_2, \dots, x_n)$$

(where λ_i are constant multipliers) and then try to find the extrema of the function w . To find stationary values of w we have to solve the system of equations

$$\frac{\partial w}{\partial x_j} = 0, \quad j = 1, 2, \dots, n$$

along with the equations of the constraints, viz.

$$\phi_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, m, \quad (m < n)$$

Both these equations involve $m+n$ unknowns $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m$.

Constrained variational problems are similar to the above problems in ordinary calculus.

9.7 The Euler-Lagrange Equation for Constrained Extrema

The following theorem states the relevant result.

Theorem

The extremal curves $y_i = y_i(x)$, $i = 1, 2, \dots, n$ of the functional

$$I[y_1(x), y_2(x), \dots, y_n(x)] = \int_{x_1}^{x_2} F(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx$$

with constraints

$$G_j(x, y_1, \dots, y_n) = \text{constant}, \quad j = 1, 2, \dots, m$$

satisfy the Euler-Lagrange equations corresponding to the functional

$$\begin{aligned} H[y_1, y_2, \dots, y_n] &= \int_{x_1}^{x_2} \left[F(x, y_1, y_2, \dots, y_n) + \sum_{i=1}^m \lambda_i(x) G_i(x) \right] dx \\ &= \int_{x_1}^{x_2} H(x, y_1, \dots, y_n) dx \end{aligned}$$

where $\lambda_i(x)$ are suitably chosen multipliers.

It is clear that the Euler-Lagrange equation in this case will be

$$\frac{\partial H}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial H}{\partial y'_i} \right) = 0, \quad i = 1, 2, \dots, n \quad (9.7.1)$$

The curves $y_i = y_i(x)$, $i = 1, 2, \dots, n$ be determined from equations (9.7.1) and the equations of the constraints, viz.

$$G_j(x, y_1, y_2, \dots, y_n) = 0, \quad j = 1, 2, \dots, m$$

9.7.1 More general variational problem with constraints

In this case we have to find the extremal curves $y = y(x)$ which extremizes

$$I[y] = \int_{x_1}^{x_2} F(x, y, y') dx$$

with endpoint conditions $y(x_1) = y_1$, $y(x_2) = y_2$ subject to

$$J[y] = \int_{x_1}^{x_2} G(x, y, y') dx = \text{constant}$$

We assume (as in the case without constraint) that F and G have continuous second order derivatives w.r.t. their arguments; similarly y is supposed to have second order continuous derivative. We consider a 2-parameter family of curves represented by

$$y(x, \epsilon_1, \epsilon_2) = y(x) + \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x)$$

The end point conditions on the curves $y(x, \epsilon_1, \epsilon_2)$ and $y(x)$ require that

$$\eta_1(x_i) = 0 = \eta_2(x_i), \quad i = 1, 2$$

Because of dependence of y on ϵ_1 and ϵ_2 , we have

$$I(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} F(x, y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, y' + \epsilon_1 \eta'_1 + \epsilon_2 \eta'_2) dx$$

$$J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} G(x, y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, y' + \epsilon_1 \eta'_1 + \epsilon_2 \eta'_2) dx = \text{constant}$$

As we vary ϵ_1 and ϵ_2 , the function $I(\epsilon_1, \epsilon_2)$ takes different values but changes in ϵ_1 and ϵ_2 conspire to keep $J(\epsilon_1, \epsilon_2)$ to the constant value of k . (This will not be possible in case of a single parameter).

We suppose that the stationary value of $I(\epsilon_1, \epsilon_2)$ corresponds to $\epsilon_1 = \epsilon_2 = 0$. Hence we must have

$$\frac{\partial I}{\partial \epsilon_1} \Big|_{\epsilon_1=\epsilon_2=0} = 0 = \frac{\partial I}{\partial \epsilon_2} \Big|_{\epsilon_1=\epsilon_2=0}$$

and

$$J(\epsilon_1 = 0, \epsilon_2 = 0) = k$$

This is equivalent to a problem in the calculus of constrained extrema in which we have to extremize the function

$$I(\epsilon_1, \epsilon_2) \text{ subject to } J(\epsilon_1, \epsilon_2) = k.$$

Since the solution corresponds to $\epsilon_1 = \epsilon_2 = 0$, we must have

$$\left(\frac{\partial I}{\partial \epsilon_1} + \lambda \frac{\partial J}{\partial \epsilon_1} \right) \Big|_{\epsilon_1=\epsilon_2=0} = 0, \quad \left(\frac{\partial I}{\partial \epsilon_2} + \lambda \frac{\partial J}{\partial \epsilon_2} \right) \Big|_{\epsilon_1=\epsilon_2=0} = 0$$

and

$$(J - k) \Big|_{\epsilon_1=\epsilon_2=0} = 0$$

The first equation is equivalent to

$$\begin{aligned} & \frac{\partial}{\partial \epsilon_1} \int_{x_1}^{x_2} [F(x, y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, y' + \epsilon_1 \eta'_1 + \epsilon_2 \eta'_2) \\ & + \lambda G(x, y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, y' + \epsilon_1 \eta'_1 + \epsilon_2 \eta'_2)] dx \Big|_{\epsilon_1=0=\epsilon_2} = 0 \end{aligned}$$

or

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta_1 + \frac{\partial F}{\partial y'} \eta'_1 + \lambda \left(\frac{\partial G}{\partial y} \eta_1 + \frac{\partial G}{\partial y'} \eta'_1 \right) \right] dx = 0$$

On integrating the second and the fourth terms by parts, and using the endpoint conditions $\eta_1(x_1) = 0 = \eta_1(x_2)$, we obtain

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \right] \eta_1(x) dx = 0$$

Similarly we have

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \right] \eta_2(x) dx = 0$$

Since

$$\frac{\partial G}{\partial y} - \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) \neq 0$$

in general, (because the functional J is not an extremum for $\epsilon_1 = \epsilon_2 = 0$ we can choose $\eta_1(x)$ such that

$$\int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \eta_1(x) dx \neq 0$$

which is always possible when

$$\frac{\partial G}{\partial y} - \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) \neq 0$$

This relation can be used to define λ . Using this value of λ in equation (2), we obtain

$$\int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \right] \eta_2(x) dx = 0$$

where $\eta_2(x)$ is arbitrary function which vanishes at the end-points. Invoking the fundamental theorem of the calculus of variations, we have the necessary condition

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \lambda \left[\frac{\partial G}{\partial y} - \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) \right] = 0$$

or

$$\frac{\partial H}{\partial y} - \frac{d}{dx} \frac{\partial H}{\partial y'} = 0$$

which is the Euler-Lagrange equation for

$$H = F + \lambda G$$

with the end-point conditions $y(x_1) = y_1$, $y(x_2) = y_2$. In actual calculations, λ is determined from the side condition

$$\int_{x_1}^{x_2} G(x, y, y') dx = k, \text{ a constant}$$