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Extension of the Euler-Lagrange equation with one independent variable

We discuss two cases of extensions of Euler-Lagrange equation.

- i) The case $F = F(x, y_1, y_2, y_3, \dots, y_n, y'_1, y'_2, \dots, y'_n)$
i.e. when there is one independent variable and n dependent variables
- ii) The case $F = F(x, y, y', y'', \dots, y^{(n)})$
i.e. the integrand in the functional involves higher order derivatives.

Euler-Lagrange equation, one independent, many dependent variables

Here the functional is of the form

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \dots, y_n, y'_1, y'_2, y'_3, \dots, y'_n) dx$$

with the boundary conditions

$$y_k(x_1) = \text{constant} \quad , \quad y_k(x_2) = \text{constt.}, \quad k=1, 2, \dots, n$$

Now

$$\begin{aligned} \delta I &= \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} \delta y_1 + \frac{\partial F}{\partial y_2} \delta y_2 + \dots + \frac{\partial F}{\partial y_n} \delta y_n \right) \right. \\ &\quad \left. + \left(\frac{\partial F}{\partial y'_1} \delta y'_1 + \frac{\partial F}{\partial y'_2} \delta y'_2 + \dots + \frac{\partial F}{\partial y'_n} \delta y'_n \right) \right] dx \end{aligned} \quad \begin{matrix} (\text{From shorthand} \\ \text{procedure for} \\ \text{finding } \delta I) \end{matrix}$$

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} \delta y_1 + \frac{\partial F}{\partial y_1'} \delta y_1' \right) + \left(\frac{\partial F}{\partial y_2} \delta y_2 + \frac{\partial F}{\partial y_2'} \delta y_2' \right) + \dots + \left(\frac{\partial F}{\partial y_n} \delta y_n + \frac{\partial F}{\partial y_n'} \delta y_n' \right) \right] dx$$

Consider

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k} \delta y_k' \right) dx = \frac{\partial F}{\partial y_k} \delta y_k \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y_k \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) dx \quad \rightarrow (1)$$

$$= - \int_{x_1}^{x_2} \delta y_k \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) dx \quad \delta y_k = 0 \quad \text{at } x = x_1, x_2$$

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k} \delta y_k + \frac{\partial F}{\partial y_k'} \delta y_k' \right) dx = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} \delta y_k - \delta y_k \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] dx \\ = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k dx \quad \rightarrow (2)$$

$$k = 1, 2, \dots, n$$

Thus

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} - \frac{d}{dx} \frac{\partial F}{\partial y_1'} \right) \delta y_1 + \left(\frac{\partial F}{\partial y_2} - \frac{d}{dx} \frac{\partial F}{\partial y_2'} \right) \delta y_2 \right. \\ \left. + \dots + \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx} \frac{\partial F}{\partial y_n'} \right) \delta y_n \right] dx \quad \rightarrow (3)$$

For extremal $\delta I = 0$

$$\int_{x_1}^{x_2} \sum_k \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k dx = 0 \quad k = 1, 2, \dots, n \quad \rightarrow (4)$$

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Since the curves (functions) are independent of each other, each of the integral on L.H.S of eq. (4), must be zero

Therefore

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_k} \right) \right] S y_k dx = 0, \quad k=1, 2, \dots$$

From fundamental theorem of calculus of variations, we have

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_k} \right) = 0 \quad k=1, 2, \dots$$

which are required equations for the extremal curves. $\rightarrow (5)$

Euler - Lagrange equations with higher order derivatives.

Consider

$$L = \int F(x, y, y', y'', \dots, y^{(n)}) dx$$

with end point conditions

$$y(x_1) = y'(x_1) = y''(x_1) = \dots = y^{(n-1)}(x_1) = \text{const}$$

$$\text{and } y(x_2) = y'(x_2) = y''(x_2) = \dots = y^{(n-1)}(x_2) = \text{constant}$$

Now

$$S^I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'} s_y' + \frac{\partial F}{\partial y''} s_y'' + \dots + \frac{\partial F}{\partial y^n} s_y^n \right) dx$$

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Consider

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} s_y' dx &= \left. \frac{\partial F}{\partial y'} s_y \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} s_y \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) dx \\ &= - \int_{x_1}^{x_2} s_y \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) dx \quad s_y = 0 \text{ at } x_1, x_2 \end{aligned}$$

Similarly

$$\begin{aligned} \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y''} s_y'' \right) dx &= \left. \frac{\partial F}{\partial y''} s_y' \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} s_y' \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) dx \\ &= - \int_{x_1}^{x_2} s_y' \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) dx \\ &= - \left. s_y \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \right|_{x_1}^{x_2} + (-1)^2 \int_{x_1}^{x_2} s_y \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) dx \\ &= (-1)^2 \int_{x_1}^{x_2} s_y \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) dx \end{aligned}$$

Similarly

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'''} s_y''' \right) dx = (-1)^3 \int_{x_1}^{x_2} s_y \frac{d^3}{dx^3} \left(\frac{\partial F}{\partial y'''} \right) dx$$

⋮

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y^{(n)}} s_y^{(n)} \right) dx = (-1)^n \int_{x_1}^{x_2} s_y \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) dx$$

Therefore (1) becomes

$$\delta F = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \delta y + (-1)^1 \delta y \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + (-1)^2 \delta y \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \right. \\ \left. + (-1)^3 \delta y \frac{d^3}{dx^3} \left(\frac{\partial F}{\partial y'''} \right) + (-1)^n \delta y \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) \right] dx$$

For extremal curve $\delta F = 0$, so

$$0 = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} + (-1) \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + (-1)^2 \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \right. \\ \left. + (-1)^3 \frac{d^3}{dx^3} \left(\frac{\partial F}{\partial y'''} \right) + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) \right] \delta y dx$$

By fundamental theorem of calculus of variations

$$\cdot \frac{\partial F}{\partial y} + (-1) \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + (-1)^2 \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \\ + (-1)^3 \frac{d^3}{dx^3} \left(\frac{\partial F}{\partial y'''} \right) + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) = 0$$

which is Euler - Lagrange equation in generalized form.

Example 1

Find the extremal for

$$L = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz) dx$$

where the B.C. are $y(0)=0$, $y(\pi/2)=1$,

$$z(0)=0, z(\pi/2)=-1$$

Solution

There are two dependent variables y and z , so using Eqs. (5), we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \text{--- (1)}$$

$$\text{and } \frac{\partial F}{\partial z} - \frac{d}{dx} \left(\frac{\partial F}{\partial z'} \right) = 0 \quad \text{--- (2)}$$

$$\text{Since } F = y'^2 + z'^2 + 2yz$$

$$\text{Therefore } \frac{\partial F}{\partial y} = 2z, \frac{\partial F}{\partial y'} = 2y', \frac{\partial F}{\partial z} = 2y, \frac{\partial F}{\partial z'} = 2z'$$

Substituting in above equations

$$2z - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow z - y'' = 0 \quad \text{--- (3)}$$

$$\text{and } 2y - \frac{d}{dx}(2z') = 0$$

$$\Rightarrow y - z'' = 0 \quad \text{--- (4)}$$

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From (3) & (4), we have

$$z'' = y''' , y = z''$$

From where

$$\therefore y''' = y''' = y$$

$$\text{or } y''' - y = 0 \quad \text{--- (5)}$$

$$\text{or } D^4 - 1 = 0 \quad (\text{Auxiliary eq. of (5)})$$

$$\text{or } (D^2 - 1)(D^2 + 1) = 0$$

$$\text{or } (D+1)(D-1)(D^2 + 1) = 0$$

$$\text{Roots are } D = -1, 1, \pm i$$

So solution is

$$y = Ae^x + Be^{-x} + C\cos x + E\sin x$$

$$\text{Also } z = y'' = \frac{d^2}{dx^2} (Ae^x + Be^{-x} + C\cos x + E\sin x)$$

$$= \frac{d}{dx} (Ae^x - Be^{-x} - C\sin x + E\cos x)$$

$$z = Ae^x + Be^{-x} - C\sin x - E\cos x$$

Applying B.C.

$$y(0) = 0$$

$$\Rightarrow 0 = A + B + C \quad \text{--- i,}$$

$$y(\gamma_2) = 1$$

$$\Rightarrow 1 = Ae^{\gamma_2} + Be^{-\gamma_2} + E \quad \text{--- ii,}$$

Similarly

$$Z(0) = 0$$

$$\Rightarrow 0 = A + B - C \quad \text{--- (iii)}$$

$$Z(\pi/2) = -1$$

$$\Rightarrow -1 = Ae^{\pi/2} + Be^{-\pi/2} - E \quad \text{--- (iv)}$$

Adding (i), & (iii),

$$A + B = 0 \quad \text{or} \quad \boxed{B = -A}$$

Also subtracting (iii), from (i), gives

$$\boxed{C=0}$$

Now adding (ii), & (iv),

$$Ae^{\pi/2} + Be^{-\pi/2} = 0 \quad \text{--- (v)}$$

and subtracting (iv), from (ii), gives

$$2 = 2E \Rightarrow \boxed{E=1}$$

Using $B = -A$ in (v),

$$Ae^{\pi/2} - Ae^{-\pi/2} = 0$$

$$\Rightarrow A(e^{\pi/2} - e^{-\pi/2}) = 0$$

$$\boxed{A=0}$$

$$\because e^{\pi/2} - e^{-\pi/2} \neq 0$$

$$\Rightarrow \boxed{B=0}$$

Thus

$$y = \sin x, \quad z = -\sin x$$