Example Give the geometrical interpretation of the variable problem

$$\int_0^1 \sqrt{1+y'^2}\,dx \rightarrow \text{minimum}$$

with the boundary conditions $y(0)=0$. and $y(1)=1$

Solve the problem for the extremal. Find the stationary value of the integral and compare it with the values of the integral which are obtained for curves that join the same end points but are different from the extremal.

Solution

(i) We have

$$ds = \sqrt{1+y'^2}\,dx$$

Given problem can be stated as

$$\int_{(0,0)}^{(1,1)} ds \rightarrow \text{minimum}$$

i.e finding the curve of shortest length through the points $(0,0)$ to $(1,1)$; which is a straight line.

(ii) Stationary value $= \int_0^1 \sqrt{1+y'^2}\, dx$

From example (i),

Using B.C. $y = \pm ax + b$

$y(0) = 0 \Rightarrow b = 0$.

$y(1) = 1 \Rightarrow 1 = \pm a(1) \Rightarrow a = \pm 1$

Therefore $y = \pm x$.

$y' = \pm 1$.

Thus

Stationary value $= \int_0^1 \sqrt{1+1}\, dx$

$= \sqrt{2}\, x \Big|_0^1 = \sqrt{2}$.

(iii) For values of the integral which are obtained for curves that join the same end points and are different from the extremal, we have for

$y = x^2$, $y' = 2x$.

$$I = \int_0^1 \sqrt{1+y'^2}\, dx$$

$$= \int_0^1 \sqrt{1+4x^2}\, dx = \frac{1}{2}\int_0^1 \sqrt{1+(2x)^2}\, 2\,dx$$

Using

$$\int_a^b \sqrt{x^2+a^2}\, dx = \left(\frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \ln\left(x+\sqrt{x^2+a^2}\right)\right]_a^b$$

$$= \frac{1}{2}\left[\frac{1}{2}2x\sqrt{1+4x^2} + \frac{1}{2}\ln\left(2x+\sqrt{1+4x^2}\right)\right]\Big|_0^1$$

$$= \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right]$$

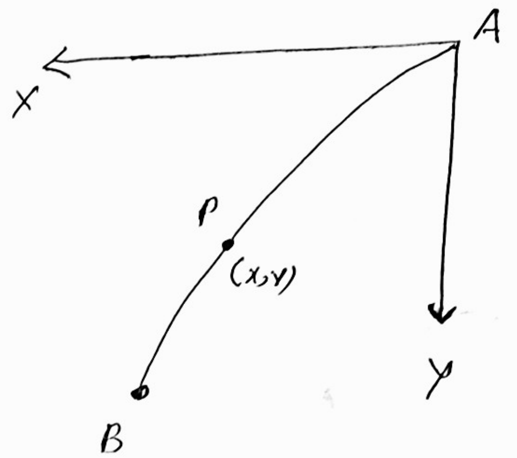
$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2+\sqrt{5}\right)$$

<u>Comparison</u> This value is always greater than $\sqrt{2}$. Which shows that $\sqrt{2}$ is the stationary value. If we take any other curve, its value will always be greater than $\sqrt{2}$

<u>Example 3</u> Find the equation of the path in space down which a particle will fall from one point to another in the shortest possible time.

<u>Solution</u>

Let a particle fall from a point A to another point B. There are infinite number of ~~paths~~ paths A and B, but we are to consider that path only along which the time taken is minimum.



We choose coordinate axes as shown in fig.

Let (x,y) be the position of the particle at time t. If ds denotes the arc element of the curve $y=y(x)$, then total time taken by the particle in falling from A to B is given by

$$T = \int_A^B dt.$$

$$= \int_A^B \frac{ds}{v} \qquad \because v = \frac{ds}{dt}.$$

$$= \int_A^B \frac{ds}{\sqrt{2gy}} \qquad v^2 = u^2 + 2as$$

$$= \frac{1}{\sqrt{2g}} \int_A^B \frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

To find extremal curve through A and B, it must satisfy the Euler-Lagrange equation

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

Here $F = \sqrt{1+y'^2} / \sqrt{y}$

Therefore

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}} \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{y(1+y'^2)}}$$

$$\therefore F - y' \frac{\partial F}{\partial y'} = a \quad (\text{constant})$$

$$\Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \cdot \frac{y'}{\sqrt{y(1+y'^2)}} = a$$

or $$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = a$$

or $$\frac{1+y'^2 - y'^2}{\sqrt{y(1+y'^2)}} = a$$

or $$\frac{1}{y(1+y'^2)} = a^2 \Rightarrow a^2(1+y'^2) = \frac{1}{y}$$

or $$1+y'^2 = \frac{1}{a^2 y} \Rightarrow y'^2 = \frac{1}{a^2 y} - 1.$$

$$\Rightarrow y'^2 = \frac{1-a^2 y}{a^2 y}.$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{1-a^2 y}{a^2 y}}$$

or $$dx = \sqrt{\frac{a^2 y}{1-a^2 y}}\, dy$$

Let $a^2 y = \sin^2 \theta/2$

Then $dy = 2 \frac{1}{a^2} \sin \theta/2 \cos \theta/2 \cdot \frac{1}{2} d\theta$

Therefore

$$dx = \sqrt{\frac{\sin^2\theta/2}{1-\sin^2\theta/2}} \frac{1}{a^2} \sin \theta/2 \cos \theta/2 \, d\theta$$

$$= \frac{1}{a^2} \frac{\sin\theta/2}{\cos\theta/2} \sin\theta/2 \cos\theta/2 \, d\theta$$

$$dx = \frac{1}{a^2} \sin^2\theta/2 \, d\theta$$

$$= \frac{1}{2a^2} (1-\cos\theta) \, d\theta$$

Integrating, we get

$$x = \frac{1}{2a^2}(\theta - \sin\theta) + b$$

Thus $x = \frac{1}{2a^2}(\theta - \sin\theta) + b, \quad y = \frac{1}{2a^2}(1-\cos\theta)$

$x = r(\theta - \sin\theta)$
$y = r(1-\cos\theta)$

These are parametric eqs. of cycloid. So the curve down which the particle takes the minimum time is cycloid.

Example 4

Find the curve joining the points (x_1, y_1) and (x_2, y_2) which gives minimum area of surface of revolution generated around (i) y-axis, (ii) x-axis.

Solution

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in xy- plane. We want to find a curve which gives the minimum area of the surface of revolution.

(i) Let us consider the case when the curve revolves about the y - axis. In this case, area of the surface of revolution will be given by

$$\begin{aligned}\text{Area} &= \int_A^B 2\pi\, x\, ds \\ &= 2\pi \int_A^B x\, ds = 2\pi \int_A^B x\, \sqrt{1+y'^2}\, dx\end{aligned}$$

For the minimum value it must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which in this case is equivalent to (because F is independent of y)

$$\frac{\partial F}{\partial y'} = \text{constant}, \quad a, \text{ say} \tag{1}$$

Here $F = x\sqrt{1+y'^2}$. Therefore

$$\frac{\partial F}{\partial y'} = x\frac{2y}{2\sqrt{1+y'^2}} = \frac{xy'}{\sqrt{1+y'^2}}$$

Putting in (1)

$$\frac{xy'}{\sqrt{1+y'^2}} = a$$

or on simplification

$$y' = \frac{a}{\sqrt{x^2-a^2}}$$

Therefore on integration by separating the variables, we have

$$\int dy = \int \frac{a}{\sqrt{x^2-a^2}}dx \quad \text{or} \quad y = a\cosh^{-1}\frac{x}{a} + c$$

(ii)

$$\text{Area} = \int_A^B 2\pi\, y\, ds = 2\pi \int_A^B y\sqrt{1+y'^2}\, dx$$

Since we want a curve which gives minimum area of the surface of revolution generated about the x - axis, so it must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which (because of no explicit dependence on x) is equivalent to

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

In this case $F = y\sqrt{1+y'^2}$. Therefore

$$\frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

The Euler-Lagrange equation becomes

$$y\sqrt{1+y'^2} - \frac{yy'}{\sqrt{1+y'^2}} = a, \text{ say}$$

or on simplification

$$\frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{a}$$

On integration

$$\int dx = \int \frac{a}{\sqrt{y^2-a^2}}\,dy \quad \text{or} \quad x = a\cosh^{-1}\frac{y}{a} + c$$

Example 5

On what curves can the functional $I = \int_0^{\pi/2}(y'^2 - y^2)\,dx$ with endpoint conditions $y(0) = 0$, $y(\pi/2) = 1$ be extremized.

Solution

Here $F = y'^2 - y^2$. The E.L. equation is given by

$$-2y - \frac{d}{dx}(2y') = 0 \text{ or } y + y'' = 0$$

whose solution can be written as

$$y = A\cos x + B\sin x$$

Now the B.C. $y(0) = 0$ gives $A = 0$. Therefore $y = B\sin x$.

Next we apply the second B.C. *viz.* $y(\pi/2) = 1$, which gives $B = 1$. Hence the required extremal is $y = \sin x$.