<u>Theorem</u> Let $I = \int_{x_1}^{x_2} F(x, y, y')\,dx$, where $y = y(x)$ is a continuous fn. having continuous first and second order derivatives satisfying the following end point conditions

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2$$

If F is supposed to have continuous first and second order derivatives w.r.t its arguments, then the function $y(x)$ will extremise the given integral if it satisfies the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

PROOF

(For Extremal curve)

Consider two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$.

There are infinite number of curves giving infinite number of values to the functional I.

Our purpose is to find the curve which gives stationary values of the functional.

Let the family of curves passing through fixed points A and B be defined as

$$y(x, \alpha) = y(x, 0) + \alpha\, \eta(x) \longrightarrow (1)$$

where $\eta(x)$ denotes the deviation from the curve $y = y(x) = y(x, 0)$ and α is the parameter labeling different paths, and is independent of x.

Let extremal curve corresponds to the value $\alpha = 0$

Since all curves pass through the end points (x_1, y_1) and (x_2, y_2). Then

$$\eta(x_1) = \eta(x_2) = 0.$$

Since $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx \longrightarrow (2)$$

From (1),

$$\frac{\partial y(x, \alpha)}{\partial \alpha} = \eta(x)$$

and $y'(x,\alpha) = y'(x, 0) + \alpha \eta'(x)$

$$\frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[y'(x,0) + \alpha \eta'(x) \right]$$

$$= \frac{\partial y'(x,0)}{\partial \alpha} + \frac{\partial}{\partial \alpha} \alpha \eta'(x)$$

$$= \eta'(x)$$

Therefore (2) becomes

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\eta(x) \frac{\partial F}{\partial y} + \eta'(x) \frac{\partial F}{\partial y'} \right] dx \longrightarrow (3)$$

Now $\int_{x_1}^{x_2} \eta'(x) \frac{\partial F}{\partial y'} dx = \eta(x) \frac{\partial F}{\partial y'} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) dx$

$$= - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) dx$$

Therefore (3) becomes

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \eta(x) \frac{\partial F}{\partial y} dx - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x) \, dx .$$

For extreme value $\frac{\partial I}{\partial \alpha} = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x) \, dx = 0 . \quad \text{--- (4)}$$

Since $\eta(x)$ is arbitrary fn. of x and $\eta(x_1) = \eta(x_2) = 0$

Further expression with in square brackets of (1) is a continuous fn. of x.

Therefore by fundamental theorem of calculus

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 . \quad \text{--- (5)}$$

Eq. (5) is known as Euler_Lagrange equation.

To show that Euler-Lagrange eq. is a second order ODE in y

Since $F = F(x, y, y')$

Therefore $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y'}$ are also fns. of x, y, y'

By chain rule

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left[\frac{\partial F(x, y, y')}{\partial y'}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}$$

$$= \frac{\partial^2 F}{\partial x \partial y'} + \frac{\partial^2 F}{\partial y \partial y'}\frac{dy}{dx} + \frac{\partial^2 F}{\partial y'^2}\frac{dy'}{dx}$$

$$= F_{xy'} + F_{yy'}\, y' + F_{y'y'}\, y''$$

Therefore (5) becomes

$$F_y - F_{xy'} + F_{yy'}\, y' + F_{y'y'}\, y'' = 0$$

or $F_{xy'} + F_{yy'}\, y' + F_{y'y'}\, y'' = F_y$

which is second order ODE.

Special Cases

1. If F is independent of y, then from Eq. (5), we have

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = \text{constant}.$$

2. If F is independent of x

Then

$$\frac{\partial F}{\partial x} = 0$$

In this case (5) becomes

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}$$

$$= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$$

$$\frac{\partial F}{\partial y} = y'\frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)$$

or $$\left(\frac{\partial F}{\partial y}\right)dy = y'd\left(\frac{\partial F}{\partial y'}\right) \longrightarrow (6)$$

Now $F = F(y, y')$

$$dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy' \longrightarrow (7)$$

Using Eq. (6) in (7)

$$dF = y'd\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}dy'$$

$$dF = d\left(y'\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow d\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

Short hand procedure for obtaining variation of the functional.

From calculus, we have

If $y = f(x)$, then

$$\delta y \approx f'(x)\, dx$$

Therefore from

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\eta(x) \frac{\partial F}{\partial y} + \eta'(x) \frac{\partial F}{\partial y'} \right] dx$$

we can obtain

$$\delta I \approx \left(\frac{dI}{d\alpha} \right) \delta\alpha$$

where $\delta\alpha$ is an increment in parameter α and will correspond to neighbouring curve.

So

$$\frac{\partial I}{\partial \alpha} \delta\alpha \equiv \delta I = \int_{x_1}^{x_2} \left[\eta(x) \frac{\partial F}{\partial y} + \eta'(x) \frac{\partial F}{\partial y'} \right] \delta\alpha\, dx$$

Using Eqs. (1) & (2),

$$y(x,\alpha) - y(x,0) \equiv \delta y \approx \delta\alpha\, \eta(x)$$

$$y'(x,\alpha) - y'(x,0) \equiv \delta y' \approx \delta\alpha\, \eta'(x)$$

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx$$

Example 1 Use calculus of variations to prove that a straight line is shortest distance between two points in a plane.

Solution

Element of length along a curve $y=y(x)$ is given by

$$ds = \sqrt{x^2+y^2}$$

$$\text{or} \quad ds = \sqrt{1+y'^2}\, dx$$

We have to minimize $I = \int_a^b \sqrt{1+y'^2}\, dx$

Subject to endpoint conditions $y(a)=y_0$, $y(b)=y_1$

Hence $F = \sqrt{1+y'^2}$

I will be minimize if $y=y(x)$ satisfies the Euler-Lagrange eq. (5),

From (5)

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

In our case

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = c$$

$$\text{or} \quad \frac{y'}{\sqrt{1+y'^2}} = c \quad \text{or} \quad y' = c\sqrt{1+y'^2}$$

$$y'^2 = c^2(1+y'^2) \Rightarrow y'^2(1-c^2) = c^2 \Rightarrow y'^2 = \frac{c^2}{1-c^2} = a^2$$

On integration $y = \pm ax + b$

which is equation of straight line.