# Chapter 9

## VARIATIONAL METHODS

In the subject of variational methods, we determine the extreme value of a functional.

### FUNCTIONAL

Let M be the set of functions defined over the interval [a,b]. If there is a rule which assigns each function of M to a real number J, then J is called the functional from M to $\mathbb{R}$.

### EXTREMAL

Extremal is the curve along which the functional takes the stationary values.

### STATIONARY VALUE

The maximum or minimum value of a function or functional is called stationary value.

### SOME EXAMPLES OF VARIATIONAL PROBLEMS

Example 1 To find the curve whose distance between two points on a surface is minimum.

## Discussion

This problem is called geodesic problem. Let A and B be any points on a curve C, lying on a surface S. The equation of surface is $z = z(x, y)$.

The distance between the points A and B on any curve $y = f(x)$ is given by

$$l = \int_A^B ds$$

$$= \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

$$= \int_A^B \sqrt{1 + y'^2 + z'^2}\, dx$$

Here ds is element of arc along the curve.

In calculus of variations, we have to find minimum value of $l$.

When points A and B lie in the XY-plane, then expression for $l$ takes the form

$$l = \int_A^B \sqrt{1 + y'^2}\, dx$$

Example 2 A particle falls under gravity from A to B. Determine the curve along which time taken by the particle will be minimum.

Discussion

To formulate this problem as a variational problem, we know that

$$v = \frac{ds}{dt}$$

$$\text{Total time Taken} = \int_A^B dt$$

$$= \int_A^B \frac{dt}{ds}\, ds$$

$$= \int_A^B \frac{1}{v}\, ds$$

$$= \int_A^B \frac{1}{\sqrt{2gy}}\, ds$$

$$= \int_A^B \frac{1}{\sqrt{2gy}} \sqrt{1+y'^2}\, dx$$

$$= \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+y'^2}{y}}\, dx$$

By minimizing this functional, we can find the required time.

<u>Example 3</u> To find the curve $y = f(x)$ which has a given length $l$ and encloses maximum area with, say, X-axis.

<u>Discussion</u>

Since

$$\text{Area} = \int_a^b f(x)\,dx \longrightarrow (1) \quad \left(\begin{array}{l}\text{Area enclosed by} \\ \text{the curve } y = f(x) \text{ b/w} \\ \text{the lines } x = a \text{ and } x = b\end{array}\right)$$

and $$l = \int_a^b ds = \int_a^b \sqrt{1 + y'^2}\,dx \longrightarrow (2)$$

Thus problem reduces to that of maximizing the area in eq. (1), subject to the condition given in (2).

<u>EULER - LAGRANGE EQUATION</u>

~~Ex~~ We will derive the relevant differential equations from which the required extremal curves or surfaces can be obtained. In each case, we have to extremize a functional of the form

$$J[y(x)] = \int_c F(x, y, y', \dots)\,dx$$

or $$J[u(x, y, \dots)] = \int_c F(x, y, \dots, u, u_x, u_y, \dots)\,dx\,dy \dots$$

The integrand $F$ have different forms in different situations. Simplest form is

$$F = F(x, y, y')$$

## Theorem

Let $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

where $y = y(x)$ is a continuous fn. having continuous first and second order derivatives satisfying the B.Cs.

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2$$

If $F$ is supposed to have continuous first and second order derivatives w.r.t its arguments, then the fn. $y(x)$ will extremise the given integral if it satisfies the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

## FUNDAMENTAL THEOREM OF CALCULUS OF VARIATIONS

### Statement

If $f(x)$ is continuous in the interval $(x_1, x_2)$ and the integral $\int_{x_1}^{x_2} f(x) g(x)\, dx$ is identically zero i.e

$$\int_{x_1}^{x_2} f(x)\, g(x)\, dx \equiv 0$$

where $g(x)$ satisfies the conditions

(i) It is an arbitrary fn. with continuous derivatives in the interval $(x_1, x_2)$.

(ii) $g(x_1) = g(x_2) = 0$

then $f(x) \equiv 0 \quad \forall \; x \in [x_1, x_2]$

Proof

Let $f(x) \neq 0$ in $(x_1, x_2)$

then $f(x_0) \neq 0$ for at least one point $x_0 \in (x_1, x_2)$

Since $f(x)$ is continuous in $(x_1, x_2)$

Then $f(x)$ is continuous at $x = x_0$

Then there exist an interval $(x_0 - \delta, x_0 + \delta)$.

where $\delta > 0$ surrounding $x_0$ such that

$f(x) > 0 \quad \forall \; x \in [x_0 - \delta, x_0 + \delta]$

Let

$$g(x) = \begin{cases} (x - x_0 + \delta)^2 (x - x_0 - \delta)^2, & x_0 - \delta \leq x \leq x_0 + \delta \\ 0 & \text{otherwise} \end{cases}$$

$g(x)$ vanishes at the end points of the interval $(x_0 - \delta, x_0 + \delta)$ and has the cont. derivative inside the interval.

then integral $\int_{x_1}^{x_2} f(x)\, g(x)\, dx$ becomes

$$\int_{x_0-\delta}^{x_0+\delta} f(x)\,(x-x_0+\delta)^2(x-x_0-\delta)^2\,dx > 0$$

This contradicts the assumption that

$$\int_{x_1}^{x_2} f(x)\,g(x)\,dx = 0$$

Hence $f(x) \equiv 0 \quad \forall\, x \in (x_1, x_2)$