METHOD OF EIGEN FUNCTION EXPANSION FOR GREEN'S FUNCTION

GREEN'S FUNCTION AND DIRAC DELTA FUNCTION

One Dimension Case

The Green's function associated with one-dimensional S-L eq.

$$L(u) + \lambda\, \delta(x)\, u = 0$$

with usual B.Cs satisfies the DE

$$L\left(G(x, x')\right) = \delta(x - x')$$

In this case soln. is

$$u(x) = \int_a^b G(x, x')\, f(x')\, dx'$$

Two Dimension Case

Consider two-dimensional PDE

$$L(u) = f(x, y) \quad ; \quad L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with homogeneous B.Cs.

then corresponding Green fn. satisfies

$$L\left[G(x|x'; y|y')\right] = \delta(x - x')\, \delta(y - y')$$

with same homogeneous B.S.

the soln. is then

$$u(x, y) = \int_c^d \int_a^b G(x|x'; y|y')\, f(x', y')\, dx' dy'$$

GREEN'S FUNCTION ASSOCIATED WITH ONE DIMENSIONAL PROBLEMS.

Consider $\quad L(u) = f(x) \longrightarrow (1)$

Subject to two homogeneous B.Cs.
Introduce the related eigenvalue problem defined by PDE.

$$L(u) = -\lambda\, r(x)\, u \longrightarrow (2)$$

Let $\quad u(x) = \sum_{n=1}^{\infty} a_n u_n(x) \longrightarrow (3)$

where $u_n(x)$ are eigenfunctions of (2) —

Substituting (3) in (1)

$$L\Big[\sum_n a_n u_n(x)\Big] = f(x)$$

or $\quad \sum_n a_n L\, u_n(x) = f(x)$

or $\quad -\sum_n a_n \lambda_n r(x)\, u_n = f(x) \;\_\_\; (4) \quad$ using (2)

$$-\sum_n a_n \lambda_n \int_a^b r(x) u_n(x) u_m(x)\,dx = \int_a^b f(x)\, u_m(x)\,dx \quad \left(\begin{array}{l}\text{Multiplying (4)}\\ \text{by } u_m(x) \text{ and}\\ \text{integrating from a to b}\end{array}\right)$$

$$-\sum_n a_n \lambda_n \delta_{m,n} = -a_m \lambda_m = \int_a^b f(x) u_m(x)\,dx$$

* where we have assumed that u_n, $n = 1, 2, \ldots$ are orthonormal w.r.t $r(x)$.

i.e $\quad \int_a^b r(x)\, u_n(x) u_m(x) = \delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

$$-a_m\lambda_m = \int_a^b f(x)\,u_m(x)\,dx$$

gives

$$\lambda_n a_n = -\int_a^b f(x)\,u_n(x)\,dx.$$

Therefore (3) becomes

$$u_p(x) = \sum_{n=1}^{\infty} -\frac{1}{\lambda_n}\int_a^b f(x')\,u_n(x')\,u_n(x)\,dx'$$

$$= \int_a^b f(x')\,G(x,x')\,dx'$$

where $G(x,x') = -\sum_n (1/\lambda_n)\,u_n(x)\,u_n(x')$.

Green's fn. does not exist if $\lambda_n = 0,\ n = 1, 2, \ldots$ i.e if any eigenvalue is zero.

<u>Example</u> Solve the problem by Green's function method.

$$u'' = f(x),\quad u(0) = 0,\quad u(l) = 0$$

<u>Solution</u>

Related homogeneous eigenvalue problem is

$$\frac{d^2u_n}{dx^2} = -\lambda u_n,\qquad u_n(0) = 0,\ u_n(l) = 0$$

or

$$\frac{d^2u_n}{dx^2} + \lambda u_n = 0$$

$$m^2 + \lambda = 0$$

$$m^2 = -\lambda \quad \text{or} \quad m = \pm i\sqrt{\lambda}$$

$$u_n(x) = A \cos \lambda_n x + B \sin \lambda_n x$$

$$u_n(0) = 0 \Rightarrow \boxed{0 = A}$$

$$u_n(l) = 0 \Rightarrow 0 = B \sin \lambda_n x$$

$B \neq 0$ (For non-trivial soln.)

$$\sin \lambda_n l = 0 = \sin n\pi \qquad n = 1, 2, \ldots$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$u_n = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

Therefore

$$G(x, x') = -\frac{2}{l} \sum_{n=1}^{\infty} \frac{(\sin n\pi x/l)(\sin n\pi x'/l)}{(n\pi/l)^2}$$

The soln. of given inhomogeneous ~~eq.~~ problem is

$$u(x) = \int_0^l f(x') \, G(x, x') dx'$$