Suppose SL operator operating on a function $y(x)$ gives as

$$\mathcal{L}\, y(x) = f(x) \longrightarrow (1)$$

which is a non-homogeneous equation.

$$\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

To solve NHE (1), define

$$\mathcal{L}\, G(x,t) = \delta(x-t)$$

So that we can show that if we define

$$y(x) = \int G(x,t)\, f(t)\, dt\,,$$

we will get Eq. (1)

<u>Proof</u>

$$\mathcal{L}\, y(x) = \mathcal{L} \int G(x,t)\, f(t)\, dt$$

Interchanging integral and differential

$$= \int \mathcal{L}\, G(x,t)\, f(x)\, dt$$

Using definition of Green's fn.

$$= \int \delta(x-t)\, f(t)\, dt$$

$$= f(x) \qquad \because \int \delta(x-t) f(t)\, dt = f(x)$$

$$\mathcal{L}\, y(x) = f(x)$$

<u>Theorem</u> If the homogeneous problem associated with the SL problem

$$\frac{d}{dx}\left(p\frac{d}{dx}\right)y + q(x)y = f(x)$$

with usual B. Cs. has trivial solution, then Green's fn. exists.

In other words if $\lambda = 0$, is not an eigen value for

$$L(y) + \lambda r(x) y = 0$$

with usual B.Cs. the Green function exists.

We have to solve the problem associated with nonhomogeneous diff. eq.

$$L\{u(x)\} + \lambda r(x) u(x) = f(x) \longrightarrow (1)$$

and $u(x)$ satisfies suitable B.Cs.

* The soln. of the nonhomogeneous diff. eq. (1), subject to B.Cs. is closely related to the existance of Green's fn.

$$L(u) + \lambda r(x) u = 0$$

If a fn. $G(x, t, \lambda)$ exists. Then soln. of (1) can be written as

$$u(x) = \int_a^b G(x, t, \lambda) f(t)\, dt.$$

$G(x, t, \lambda)$ is called Green's fn. and satisfies the eq.

$$L(G) + \lambda r(x) G = \delta(x - t)$$

Green's function associated with Regular S.L. System

Let $L(u) + \lambda r(x) u = 0 \longrightarrow$ SL eq.

$$\left.\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0 \end{aligned}\right\} \text{ end point conds.}$$

$$L = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$$

Under the assumption that $\lambda = 0$ is not an eigenvalue of above system i.e it gives trivial soln. the Green's fn. $G(x,t)$ associated with the system has the following properties

1. $G(x,t)$ satisfies the diff. eq.

$$L\{G(x,t)\} = 0 \quad \text{in each of}$$

sub intervals $[a,t)$ and $(t,b]$.

2. $G(x,t)$ is continuous for each value of x in the whole interval $[a,b]$

3. $G(x,t)$ as a fn. of x satisfies the end-point conditions.

4. $\frac{dG(x,t)}{dx}$ is discontinuous as $x \to t$ and moreover

$$\lim_{x \to t+0} G'(x,t) - \lim_{x \to t-0} G'(x,t) = -\frac{1}{p(t)}$$

Example 1

Construct Green's fn. associated with the problem

$$u'' + \lambda u = 0 \quad \text{—(1)}, \qquad u(0) = 0, \quad u(1) = 0 \quad \text{—(2)}$$

Solution

Here $p(x) = 1$, $p(t) = 1$ $\qquad \frac{d}{dx}\left(p(x)\frac{d}{dx}\right)u + q(x)u =$

Step 1 First we verify $\lambda = 0$ is an eigenvalue

$\lambda = 0$.

Eq. (1) becomes

$$u'' = 0 \Rightarrow u = Ax + B$$

Using B.Cs. (2)

$$u(0) = 0 \Rightarrow \boxed{0 = B}$$

$$u(1) = 0 \Rightarrow \boxed{0 = A}$$

Therefore $u = 0$ is the soln. of the problem corresponding $\lambda = 0$. Therefore $\lambda = 0$ is not an eigenvalue

(ii) $G(x,t)$ satisfies the differential eq.

$$\frac{d^2 G(x,t)}{dx^2} = 0$$

in each sub interval $[0, t]$ and $[t, 1]$

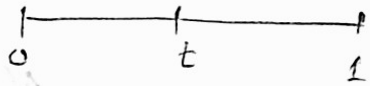
Therefore

$$G(x,t) = \begin{cases} Ax + B, & 0 \le x < t \\ A'x + B', & t < x \le 1 \end{cases} \qquad (3)$$

(iii) $G(x,t)$ is continuous everywhere and in particular at $x = t$

Therefore

$$\lim_{x \to t^+} G(x,t) = \lim_{x \to t^-} G(x,t).$$



$$A't + B' = At + B$$

or $$B' = (A - A')t + B$$

So (3) becomes

$$G(x,t) = \begin{cases} Ax + B, & 0 \le x < t \\ A'x + (A - A')t + B, & t < x \le 1 \end{cases} \qquad (4)$$

(iv) $G(x,t)$ satisfies end point conds.

$$G(0,t) = 0, \quad G(1,t) = 0 \qquad (5)$$

Eqs. (4) & (5) gives

$$\boxed{0 = B} \qquad (6)$$

and $$0 = A' + (A - A')t + 0$$

or ~~$(A - A')t = -A'$~~

$$A'(1 - t) = -At$$

or $$\boxed{A = \frac{t-1}{t} A'} \qquad (7)$$

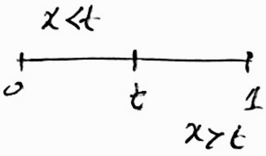
Using (6) & (7), (4) becomes

$$G(x,t) = \begin{cases} A'(t-1)\,x/t, & 0 \le x < t \\ A'x + (-A'/t)\,t, & t < x \le 1 \end{cases}$$

or
$$G(x,t) = \begin{cases} A'(t-1)\,x/t, & 0 \le x < t \\ A'x - A', & t < x \le 1 \end{cases}$$

iv) $G(x,t)$ satisfies

$$G'(t+0,t) - G'(t-0,t) = -\frac{1}{p(t)}$$



$$A' - A'(t-1)/t = -1$$

or
$$A' - A' + \frac{A'}{t} = -1$$

or
$$A' = t$$

Thus
$$G(x,t) = \begin{cases} -(t-1)\,x, & 0 \le x < t \\ -t\,(x-1), & t < x \le 1 \end{cases}$$

<u>Example 2</u> Do it yourself.

<u>Example 3</u>

Construct Green's fn. for B.V.P

$$x u'' + u' - \frac{n^2}{x} u + \lambda r(x) u = 0 \longrightarrow (1)$$

$$u(0) \text{ is finite and } u(1) = 0 \longrightarrow (2)$$

Eq. (1) can be written as

$$\left\{\frac{d}{dx}\left(x\frac{d}{dx}\right)\right\}u - \frac{n^2}{x}u + \lambda r(x)u = 0$$

Comparing with

$$L(u) + \lambda r(x)u = 0 .$$

where
$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

we get

$$p(x) = x, \quad q(x) = -\frac{n^2}{x}$$

(i) We first check if $\lambda = 0$ is an eigenvalue of homogeneous equation

$$xu'' + u' - \frac{n^2}{x}u = 0$$

$$\text{or} \quad x^2u'' + xu' - n^2u = 0, \qquad \text{(Euler's Eq.)}$$

Let $u = x^p$, then

$$u' = px^{p-1}, \quad u'' = p(p-1)x^{p-2}$$

$$x^2 p(p-1)x^{p-2} + x\,px^{p-1} - n^2x^p = 0$$

$$\text{or} \quad p(p-1)x^p + x^p p - n^2x^p = 0$$

$$\text{or} \quad [p(p-1) + p - n^2]x^p = 0$$

$$\text{or} \quad p^2 - p + p - n^2 = 0$$

$$\text{or} \quad p^2 = n^2 \Rightarrow p = \pm n$$

So general soln. is

$$u = Ax^n + Bx^{-n}$$

Using B.Cs.

$u(0)$ is finite $\Rightarrow \boxed{B=0}$

$u(1) = 0 \Rightarrow \boxed{A=0}$

Therefore $u=0$ (Trivial soln.)

So $\lambda = 0$ is not an eigenvalue. So we can associate Green's fn. to the System.

(ii) $G(x,t)$ satisfies the equation

$$xG'' + G' - \frac{n^2}{x} G = 0 \quad \text{in } [0,t) \,\&\, (t,1]$$

$$G(x,t) = \begin{cases} Ax^n + Bx^{-n}, & 0 \le x < t \\ A'x^n + B'x^{-n} & t < x \le 1. \end{cases}$$

(iii) $G(x,t)$ satisfies the given B.Cs.

$G(0,t)$ is finite & $G(1,t) = 0$

$G(0,t)$ is finite $\Rightarrow \boxed{B=0}$

$G(1,t) = 0 \Rightarrow A' + B' = 0 \Rightarrow \boxed{B' = -A'}$

Therefore

$$G(x,t) = \begin{cases} Ax^n, & 0 \le x < t \\ A'x^n - A'x^{-n}, & t < x \le 1 \end{cases}$$

(iv) $G(x,t)$ is conts at all pts. and in particular at $x=t$

$$\Rightarrow \quad G(t+0,t) = G(t-0,t)$$

$$A't^n - A't^{-n} = At^n$$

$$A'(t^n - t^{-n}) = At^n$$

$$A'\left(t^n - \frac{1}{t^n}\right) = At^n$$

$$A = \frac{t^{2n}-1}{t^n \cdot t^n}A' = \frac{t^{2n}-1}{t^{2n}}A'$$

Hence

$$G(x,t) = \begin{cases} A'x^n(1-t^{-2n}), & 0 \leq x < t \\ A'(x^n - x^{-n}), & t < x \leq 1 \end{cases}$$

(v)

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$nA't^{n-1}(1-t^{-2n}) - A'(nt^{n-1} + nt^{-n-1}) = \frac{1}{t}$$

$$nA'\left[\cancel{t^{n-1}} - t^{-n-1} - \cancel{t^{n-1}} - t^{-n-1}\right] = \frac{1}{t}$$

$$-2nt^{-n-1}A' = \frac{1}{t}$$

$$\text{or} \quad A' = -\frac{t^{n+1}}{2nt} = -\frac{t^n}{2n}$$

Hence

$$G(x,t) = \begin{cases} -t^n/2n \, x^n(1-t^{-2n}), & 0 \leq x < t \\ -\frac{t^n}{2n}(x^n - x^{-n}), & t < x \leq 1 \end{cases}$$

$$G(x,t) = \begin{cases} -x^n(t^n - t^{-n})/2n, & 0 \leq x < t \\ -t^n(x^n - x^{-n})/2n, & t < x \leq 1 \end{cases}$$