Example 1

Solve the problem

$$\frac{d^2u}{dx^2} = f(x) \;\longrightarrow (1), \quad u(0)=0, \; u(l)=0, \longrightarrow \quad 0 \leq x \leq l$$

SOLUTION

This is regular SL system with $p(x) = 1$.

$$* \quad \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda w(x) y$$

Integrating (1) w.r.t 'x' from 0 to x

$$\int_0^x \frac{d^2u}{dx^2}\,dx = \int_0^x f(x)\,dx$$

or $$\left.\frac{du}{dx}\right|_0^x = \int_0^x f(x)\,dx$$

or $$u'(x) - u'(0) = \int_0^x f(x)\,dx \longrightarrow (2)$$

Integrating (2) again

$$\int_0^x [u'(x) - u'(0)]\,dx = \int_0^x \left[\int_0^x f(x')\,dx'\right] dx$$

$$u(x)\Big|_0^x - u'(0)x\Big|_0^x = \int_0^x \int_0^{x''} f(x')\,dx'\,dx''$$

$$u(x) - u(0) - u'(0)x = \int_0^x \int_0^{x''} f(x')\,dx'\,dx''$$

Let $u'(0) = A$

$$u(x) = \int_0^x \int_0^{x''} f(x')\,dx'\,dx'' + Ax \longrightarrow (3)$$

Now
$$u(x) = \int_0^x \int_0^{x''} f(x')\,dx'\,dx'' + Ax$$

$$= \int_0^x f(x')\left[\int_{x'}^x dx''\right]dx' + Ax$$

$$= \int_0^x f(x')(x-x')\,dx' + Ax$$

$$u(x) = \int_0^x (x-x')\,f(x')\,dx' + Ax$$

Now using B.C $u(l)=0$

$$0 = \int_0^l (l-x')f(x')\,dx' + Al$$

$$\Rightarrow A = \frac{1}{l}\int_0^l (x'-l)\,f(x')\,dx'$$

Therefore
$$u(x) = \int_0^x (x-x')f(x')\,dx' \cancel{+Ax}$$
$$+ \frac{x}{l}\int_0^l (x'-l)\,f(x')\,dx' \longrightarrow (4)$$

Rewrite Eq. (4) as

$$u(x) = \int_0^x (x-x')\,f(x')\,dx'$$
$$+ \frac{x}{l}\left[\int_0^x f(x')(x'-l)\,dx' + \int_x^l f(x')(x'-l)\,dx'\right]$$

$$u(x) = \int_0^x f(x') \left[(x - x') + \frac{x}{l}(x' - l) \right] dx'$$

$$+ \int_x^l (x' - l) f(x') dx'$$

$$= \int_0^x f(x') \left[\frac{xx'}{l} - x' \right] dx' + \int_x^l (x' - l) f(x') dx'$$

$$= \int_0^x \frac{x'}{l}(x - l) f(x') dx' + \int_x^l (x' - l) f(x') dx'$$

$$u(x) = \int_0^l G(x, x') f(x') dx'$$

where
$$G(x, x') = \begin{cases} x'(x-l)/l , & 0 \leq x' \leq x \\ x(x'-l)/l , & 0 < x' \leq l \end{cases} \longrightarrow (6)$$

where $G(x, x')$ is the Green's function associated with the given problem. If we know the Green's fn. of the problem, we can find its solution.

PROPERTIES OF GREEN FUNCTION

In eq. (6) Green's fn. is function of x'. We can also write it as a fn. of x as

$$G(x, x') = \begin{cases} x(x'-l)/l , & 0 \leq x \leq x' \\ x'(x-l)/l , & x' < x \leq l \end{cases} \longrightarrow (7)$$

From (5) & (6), we deduce following properties of $G(x,x')$

1. It is symmetric in x and x', i.e

$$G(x,x') = G(x',x)$$

2. $G(x,x')$ as a fn. of x satisfies the DE $\frac{d^2}{dx^2}G(x,x') = 0$ in each of the intervals $0 \le x \le x'$ and $x' < x \le l$.

3. $G(x,x')$ as a fn. of x also satisfies $G(0,x') = 0$ and $G(l,x') = 0$, which are the same B.Cs. as satisfied by u.

4. $G(x,x')$ is cont. fn. of x in the entire interval $[0,l]$. In constructing $G(x,x')$, we will use its continuity at $x = x'$.

$$G(x'-0,x') = \frac{x'}{l}(x'-l)$$

and

$$G(x'+0,x') = \frac{x'}{l}(x'-l)$$

5. If we calculate $\frac{d}{dx}G(x,x')$, we find

$$G'(x,x') = \begin{cases} (x'-l)/l, & 0 \le x \le x' \\ x'/l, & x' < x \le l \end{cases}$$

From these relations, it follows that

$$G'(x'-0,x) - G'(x'+0,x') = 1.$$

This shows that $G'(x,x')$ as a fn. of x is discontinuous at $x = x'$.

and the discontinuity is given by (8) where $p(x') = 1$.

Example 2

Solve the problem and obtain the associated Green's function. Also discuss the properties of the Green function.

$$\frac{d^2y}{dx^2} + k^2y = f(x). \quad y(0) = 0, \quad y(\ell) = 0, \quad 0 \le x \le \ell \tag{1}$$

Solution

The given differential equation is linear nonhomogeneous DE of order 2. (with constant coefficients). Its general solution is given by

$$y = y_c + y_p$$

where y_c is the complementary function i.e. general solution of $y'' + k^2 y = 0$ and is given by

$$y_c = c_1 \cos kx + c_2 \sin kx$$

and y_p is a particular integral of the given nonhomogeneous DE. We use the method of variation of parameters to calculate y_p. By this method, we obtain

$$y_p = (\cos kx)\, u_1 + (\sin kx)\, u_2$$

where

$$u_1 = -\int_{x_0}^{x} \frac{f(x')\, y_2(x')}{W(y_1,\, y_2)(x')}\, dx'$$

and

$$u_2 = \int_{x_0}^{x} \frac{f(x')\, y_1(x')}{W(y_1,\, y_2)(x')}\, dx'$$

where x_0 is some fixed point in the domain of definition of the solution, and $W(y_1, y_2)$ is the Wronskian given by

$$W(y_1,\, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

In this problem $y_1 = \cos kx$, $y_2 = \sin kx$ and $x_0 = 0$. Therefore

$$W(y_1, y_2) = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix} = k$$

and

$$u_1(x) = -\int_0^x \frac{1}{k} f(x') \sin kx'\,dx', \quad u_2(x) = \int_0^x \frac{1}{k} f(x') \cos kx'\,dx'$$

Therefore

$$\begin{aligned} y_p &= -\frac{\cos kx}{k}\int_0^x f(x')\sin kx'\,dx' + \frac{\sin kx}{k}\int_0^x f(x')\cos kx'\,dx' \\ &= \frac{1}{k}\int_0^x f(x')\left[\sin kx\ \cos kx' - \cos kx\ \sin kx'\right] \\ &= \frac{1}{k}\int_0^x f(x')\sin k(x-x')\,dx' \end{aligned}$$

Hence the general solution of DE (1) is given by

$$\begin{aligned} &= c_1\cos kx + c_2\sin kx \\ &+ \frac{1}{k}\int_0 f(x')\sin k(x-x')\,dx \end{aligned} \qquad (2)$$

Now we apply the B.Cs. in (1). The B.C. $y(0) = 0$ gives $c_1 = 0$. Similarly the B.C. $y(\ell) = 0$ gives

$$0 = 0 + c_2\sin k\ell + \frac{1}{k}\int_0^\ell f(x')\sin k(\ell - x')\,dx'$$

or

$$c_2 = -\frac{1}{k\sin k\ell}\int_0^\ell f(x')\sin k(\ell - x')\,dx'$$

with the condition that $\sin k\ell \neq 0$.

On substituting in (2), we obtain the solution as

$$\begin{aligned} y &= -\frac{\sin kx}{k\sin k\ell}\int_0^\ell f(x')\sin k(\ell - x')\,dx' \\ &+ \frac{1}{k}\int_0^x f(x')\sin k(x-x')\,dx' \\ &= -\frac{\sin kx}{k\sin k\ell}\int_0^x f(x')\sin k(\ell - x')\,dx' \end{aligned}$$

$$
\begin{aligned}
& - \frac{\sin kx}{k \sin k\ell} \int_x^\ell f(x') \sin k(\ell - x')\, dx' \\
& + \frac{1}{k} \int_0^x f(x') \sin k(x - x')\, dx' \\
& = \frac{1}{k} \int_0^x f(x') \left[- \frac{\sin kx \sin k(l - x')}{\sin kl} + \sin k(x - x') \right] dx' \\
& - \frac{\sin kx}{k \sin kl} \int_x^l f(x') \sin k(l - x')\, dx'
\end{aligned}
$$

or on simplification

$$
\begin{aligned}
y &= \frac{1}{k} \int_0^x f(x') \left(\frac{\sin kx'}{\sin k\ell} \sin k(x - \ell) \right) dx' \\
&+ \int_x^\ell \frac{- \sin kx \sin k(\ell - x')}{k \sin k\ell} f(x')\, dx'
\end{aligned}
$$

Finally we can write

$$y = \int_0^\ell G(x,\, x')\, f(x')\, dx'$$

where $\quad 0 \le t < x$

$$G(x,\, x') = \begin{cases} \sin kx' \ \sin k(x - \ell)/(k \sin k\ell), & 0 \le x' < x \\ \sin kx \ \sin k(x' - \ell)/(k \sin k\ell), & x < x' \le \ell \end{cases} \tag{3}$$

where $G(x,\, x')$ can also be written as,

$$G(x,\, x') = \begin{cases} \sin kx \ \sin k(x' - \ell)/(k \sin k\ell), & 0 \le x < x' \\ \sin kx' \ \sin k(x - \ell)/(k \sin k\ell), & x' < x \le \ell \end{cases} \tag{4}$$

where $\sin k\ell \neq 0$ i.e. k is not an eigenvalue of the associated homogeneous problem.

Properties of the Green Function

It is easy to prove that $G(x,\, x')$ defined by (3) and (4) has the following properties:

1. It is continuous in the interval, $[0, \ell]$ and in particular at $x = x'$.
2. It satisfies the homogeneous DE

$$\left(\frac{d^2}{dx^2} + k^2 \right) G(x,\, x') = 0$$

in each of the subintervals $[0,\ x')$ and $(x',\ \ell]$.

3. $G(x,\ x')$ also satisfies the given B.Cs. $G(0,\ x') = 0$ and $G(\ell,\ x) = 0$.

4. $G'(x,\ x') \equiv \partial G(x,\ x')/\partial x$ is discontinuous at $x = x'$. This can be seen from the following:

$$G'(x,\ x') = \begin{cases} \cos kx \ \sin k(x' - \ell)/\sin k\ell, & x < x' \\ \sin kx' \ \cos k(x - \ell)/\sin k\ell, & x > x' \end{cases}$$

Therefore

$$G'(x' - 0,\ x') = \frac{\sin kx' \ \cos k(x' - \ell)}{\sin k\ell}$$

$$G'(x' + 0,\ x') = \frac{\cos kx' \ \sin k(x' - \ell)}{\sin k\ell}$$

Hence

$$\begin{aligned} G'(x' - 0,\ x') - G'(x' + 0,\ x') &= \frac{\sin kx' \cos kx' \cos k\ell + \sin kx' \sin kx'}{\sin k\ell} \\ &- \cos kx' \sin kx' \cos k\ell + \cos kx' \cos kx' \sin k\ell \\ &= \frac{\sin k\ell}{\sin k\ell}(\sin^2 kx' + \cos^2 kx') = 1 \end{aligned}$$

i.e. $G'(x' - 0,\ x') - G'(x' + 0,\ x') = 1$, where $p(x') = 1$.

8.2.1 Existence of Green's function

We have discussed the following problems and shown that Green's function exists for each of them.

$$1.\quad y'' = f(x),\ \ y(0) = 0,\ \ y(\ell) = 0$$
$$2.\quad y'' + k^2 y = f(x),\ \ y(0) = 0,\ \ y(\ell) = 0$$

We also note that the solution of the associated homogeneous problem is trivial in each case. This can be seen by first writing the general solution and then making it satisfy the B.Cs. Another way of stating this fact is to say that $\lambda = 0$ is **not** an eigenvalue of the problems

$$y'' + \lambda\, y = 0,\ \ y(0) = y(\ell) = 0$$

and

$$y'' + k^2\, y = 0,\ \ y(0) = 0,\ \ y(\ell) = 0$$

The general theorem which guarantees the existence of the Green function in such problems is stated below.

Theorem

If the homogeneous problem associated with the SL problem

$$(py')' + q(x)\,y = f(x)$$

with usual B.Cs. has trivial solution, then Green's function exists.

In other words if $\lambda = 0$, is not an eigenvalue for

$$L(y) + \lambda\, r(x)\, y = 0$$

with usual B.Cs. the Green function exists.

We have to solve the problem associated with nonhomogeneous differential equation

$$L\{u(x)\} + \lambda\, r(x)\, u(x) = f(x) \tag{8.2.1}$$

where

$$L \equiv \frac{d}{dx}\{p(x)\frac{d}{dx}\} + q(x)$$

and $u(x)$ satisfies suitable boundary conditions.

The solution of the nonhomogeneous differential equation (8.2.1) subject to B.Cs. is closely related to the existence of Green's function associated with the homogeneous equation.

$$L(u) + \lambda\, r(x)\, u = 0$$

If a function $G(x,\ t,\ \lambda)$ which does not depend on the source function $f(x)$ exists, then the solution of (8.2.1) can be written as

$$u(x) = \int_a^b G(x,\ t,\ \lambda)\ f(t)\ dt$$

$G(x,\ t,\ \lambda)$ is called *Green's function* and satisfies the equation

$$L(G) + \lambda\, r(x)\, G = \delta(x - t)$$

Chapter 8

GREEN'S FUNCTIONS AND ASSOCIATED BOUNDARY VALUE PROBLEMS

THE DIRAC DELTA FUNCTION

It can be regarded as the generalization of dirac delta function δ_{ij} which is defined as

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

and

$$\sum_j a_j \delta_{ij} = a_i$$

The Dirac delta function is defined by the following properties

(i) $$\delta(x-t) = \begin{cases} 0, & x \neq t \\ \infty, & x = \infty \end{cases}$$

(ii) $$\int_{-\infty}^{\infty} \delta(x-t)\, dx = 1$$

(iii) $$\int_{-\infty}^{\infty} \delta(x-t) f(x)\, dx = f(t)$$

Motivation for Green's function

Physically Green's function is the response corresponding to unit source. For example in electromagnetic theory the potential at a field point $\vec{r}$ due to a unit charge (unit source at a point r' is the Green function $G(r,r')$, where

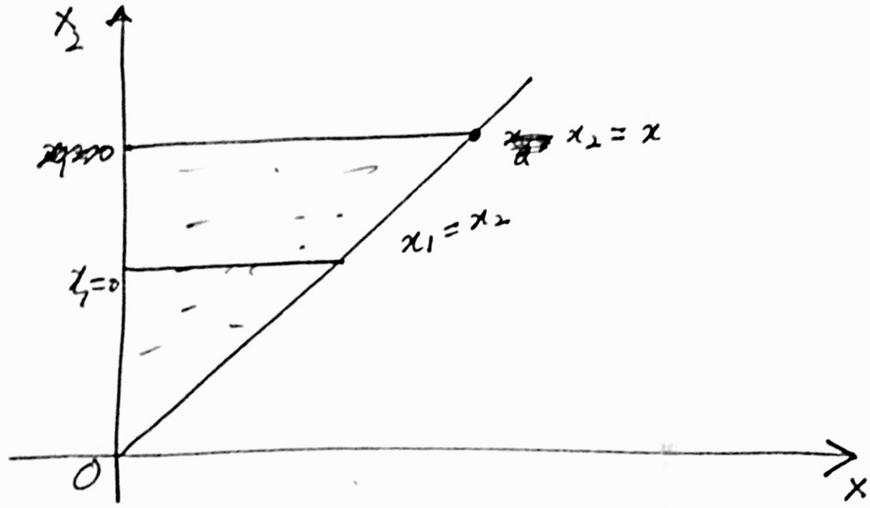
$$G(r,r') = \frac{Kq}{|r-r'|}$$

where q is the charge at the field point.

AN IMPORTANT RESULT

We will show that

$$\int_0^x \int_0^{x_2} \phi(x_1)\, dx_1\, dx_2 = \int_0^x (x - x_1)\, \phi(x_1)\, dx_1$$



Region of integration in double integral.

$$\int_0^x \int_0^{x_2} \phi(x_1)\, dx_1\, dx_2 = \int_0^x \left[\int_{x_1}^{x} dx_2 \right] \phi(x_1)\, dx_1$$

$$\int_0^x \int_0^{x_2} \phi(x_1)\, dx_1\, dx_2 = \int_0^x (x - x_1)\, \phi(x_1)\, dx_1$$