USE OF COMPLEX FOURIER TRANSFORM IN SOLVING B.V. / I.V. PROBLEMS

Example 1

Solve the problem by using Fourier transform method.

$$U_{xx}(x,t) = U_t(x,t) \quad , \quad -\infty < x < \infty \ , \ t \geq 0 \longrightarrow (1)$$

$$U(x,0) = e^{-\alpha x^2} \ , \ U(x) \text{ and } U_x(x) \to 0 \text{ as } x \to \pm\infty \longrightarrow (2)$$

SOLUTION

Taking Fourier transform of both sides of (1) w.r.t 'x', we get

$$\mathcal{F}\{U_{xx}(x,t)\} = \mathcal{F}\{U_t(x,t)\}$$

$$(-iK)^2 \mathcal{F}\{U(x,t)\} = \frac{d}{dt}\mathcal{F}\{U(x,t)\}$$

$$-K^2 U(K,t) = \frac{dU(K,t)}{dt}$$

$$\frac{dU(K,t)}{dt} \Big/ U(K,t) = -K^2$$

Integrating

$$\ln U(K,t) = -K^2 t + \text{constant}$$

$$U(K,t) = e^{-K^2 t + c_1}$$

$$U(K,t) = A e^{-K^2 t} \longrightarrow (3) \ ; \quad A = e^{c_1}$$

Taking Fourier transform of I.C

$$\mathcal{F}\{U(x,0)\} = \mathcal{F}\{e^{-\alpha x^2}\}$$

$$U(K,0) = \frac{1}{\sqrt{2\alpha}} e^{-K^2/4\alpha} \quad \left[\text{See example 1 Page 257}\right] \longrightarrow (4)$$

Using Eq. (4), in Eq. (3),

$$\boxed{\frac{1}{\sqrt{2\alpha}} e^{-K^2/4\alpha} = A}$$

So

$$U(K,t) = \frac{1}{\sqrt{2\alpha}} e^{-K^2/4\alpha} \cdot e^{-K^2 t}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-K^2\left(\frac{1}{4\alpha}+t\right)}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-\alpha' K^2} \quad ; \quad \alpha' = \frac{1+4\alpha t}{4\alpha}$$

$$u(x,t) = \mathcal{F}^{-1}\{U(K,t)\}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-iKx} \cdot e^{-\alpha' K^2} dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K^2 + \frac{ixK}{\alpha'}\right)} dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K^2 + 2K\frac{ix}{2\alpha'} + \left(\frac{ix}{2\alpha'}\right)^2 - \left(\frac{ix}{2\alpha'}\right)^2\right)} dK$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K + \frac{ix}{2\alpha'}\right)^2 + \frac{\alpha' i^2 x^2}{4\alpha'^2}} dK$$

$$= \frac{e^{-x^2/4\alpha'}}{2\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} e^{-\alpha'\left(K + \frac{ix}{2\alpha'}\right)^2} dK$$

$$= \frac{e^{-x^2/4\alpha'}}{2\sqrt{\alpha\pi}} \sqrt{\frac{\pi}{\alpha'}} = \frac{\exp\left(\frac{-x^2 \cdot 4\alpha}{4(1+4\alpha t)}\right)}{2\sqrt{\alpha}} \cdot \sqrt{\frac{4\alpha}{1+4\alpha t}}$$

$$= \exp\left(\frac{-x^2\alpha}{1+4\alpha t}\right) \frac{1}{\sqrt{1+4\alpha t}}$$

Example 3

Solve by the Fourier transform method

$$u_{xxxx} = \frac{1}{a^2} u_{tt} \longrightarrow (1)$$

where $u(x,0) = f(x)$, $u_t(x,0) = a\,g'(x)$

$g, u, u_x, u_{xx}, u_{xxx} \longrightarrow 0$ as $x \to \pm\infty$ $\longrightarrow (2)$

SOLUTION

Taking Fourier transform of (1) w.r.t 'x'

$$\mathcal{F}\left\{\frac{\partial^4 u(x,t)}{\partial x^4}\right\} = \mathcal{F}\left\{\frac{1}{a^2} u_{tt}(x,t)\right\}$$

$$(-ik)^4 U(k,t) = \frac{1}{a^2}\frac{d^2}{dt^2} U(k,t)$$

or $$\frac{1}{a^2}\frac{d^2U}{dt^2} - (-i)^4 k^4 U = 0$$

or $$\frac{1}{a^2}\frac{d^2U}{dt^2} - k^4 U = 0$$

or $$\frac{d^2U}{dt^2} - a^2k^4 U = 0 \longrightarrow (3)$$

Auxilliary Eq. of (3) is

$$D^2 - a^2k^4 = 0 \Rightarrow D^2 = a^2k^4 \quad \text{or } D = \pm ak^2$$

So $$U(k,t) = Ae^{ak^2t} + Be^{-ak^2t} \longrightarrow (*)$$

Taking Fourier transform of (2)

$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$

$U(k,t) = F(k)$ $\quad\text{—}(4)$

$\mathcal{F}\{u_t(x,0)\} = a\,g'(x)$

$$\left.\frac{d}{dt}U(k,t)\right|_{t=0} = a(-ik)G(k)$$

$$\frac{dU(k,0)}{dt} = -iak\,G(k) \quad\text{—}(5)$$

Using Eqs. (4) & (5) in Eq. (3),

$$F(K) = A + B \quad \text{——} \quad (i)$$

$$-aiKG(K) = aK^2A - aK^2B \quad \text{——} \quad (ii)$$

(ii) can be written as

$$-i\,G(K) = KA - KB \quad \text{——} \quad (iii)$$

$$KF(K) = KA + KB$$

$$2KA = KF - iG$$

$$\boxed{A = \frac{KF - iG}{2K}}$$

From (i)

$$B = F - A = F - \frac{KF - iG}{2K}$$

$$= \frac{2KF - KF + iG}{2K}$$

$$\boxed{B = \frac{KF + iG}{2K}}$$

Substituting values of A & B in Eq. (*),

$$U(K,t) = \frac{KF - iG}{2K} e^{\alpha K^2 t} + \frac{KF + iG}{2K} e^{-\alpha^2 K^2 t}$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, U(K,t)\, dK$$