7.5 Convolution and Convolution Theorem

Definition

If $f(x)$ and $g(x)$ are functions of x defined over the interval $(-\infty, +\infty)$ then their convolution, denoted by $f \star g$ is defined as follows

$$f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta) g(x-\eta)\, d\eta$$

We can show that $g \star f = f \star g$.

Now consider

$$\begin{aligned} f \star g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, g(x - \eta)\, d\eta \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x - x')\, g(x')\, dx', \quad \text{where} \quad x - \eta = x' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x')\, f(x - x')\, dx' = g \star f \end{aligned}$$

In a similar manner we can prove the following properties of the convolution.

$$f \star (g \star h) = (f \star g) \star h$$

and

$$f \star (g + h) = f \star g + f \star h$$

7.5.1 Convolution theorem

If $F(k)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$, then

$$\mathcal{F}\{f \star g\} = F(k)\, G(k)$$

or

$$\mathcal{F}^{-1}\{F(k)\, G(k)\} = f(x) \star g(x)$$

Proof

$$\begin{aligned} \mathcal{F}^{-1}\{F(k)G(k)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, F(k)G(k)\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'}\, g(x')\, dx' \right\} dk \end{aligned}$$

where we have used the definition of the Fourier transform of $g(x)$. Changing the order of integration we have

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} e^{-\imath k(x-x')}F(k)dk\right\}g(x')\,dx'$$

Now by definition

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\imath k(x-x')}\,F(k)\,dk = f(x-x')$$

We find that

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-x')\,g(x')\,dx' = f\star g$$

Hence

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = f(x)\star g(x)$$

which is equivalent to

$$F(k)G(k) = \mathcal{F}\{f\star g\}$$

7.6 Parseval's Theorems

These theorems are named after the French mathematician Marc An-otoine des Chenes Parseval (1755-1836). Note that there are similar theorems in the theory of Fourier series. These theorems are also referred to as Parseval's identities.

The Parseval first and the second theorem may be stated as follows:

$$\int_{-\infty}^{+\infty} F(k)\,\overline{F(k)}\,dk = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx \text{ or } \int_{-\infty}^{\infty}|F(k)|^2\,dk = \int_{-\infty}^{+\infty}|f(x)$$

and

$$\int_{-\infty}^{+\infty} F(k)\,G(k)\,dk = \int_{-\infty}^{+\infty} f(u)g(-u)\,du = \int_{-\infty}^{+\infty} f(x)g(-x)\,dx$$

Proof

We prove the second theorem. The first follows from it as a special case. By convolution theorem

$$\mathcal{F}^{-1}\{F(k)\,G(k)\} = f(x) \star g(x)$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} F(k)G(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)\,g(x-u)\,du$$

Putting $x = 0$ on both sides, we have

$$\int_{-\infty}^{+\infty} F(k)\,G(k)\,dk = \int_{-\infty}^{+\infty} f(u)g(-u)\,du = \int_{-\infty}^{+\infty} f(x)\,g(-x)\,dx$$

which is Parseval's second theorem. To derive the first theorem from it, we take $g(-x) = \overline{f(x)} i.e. g(x) = \overline{f(-x)}$.

Therefore

$$\mathcal{F}\{g(x)\} = \mathcal{F}\{\bar{f}(-x)\} \text{ or } G(k) = \overline{F(k)}$$

Hence

$$\int_{-\infty}^{+\infty} F(k)\,\overline{F(k)}\,dk = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx \text{ or } \int_{-\infty}^{\infty} |F(k)|^2\,dk = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

which can equivalently be written as

$$\|F\| = \|f\|$$

7.7 The Fourier Integral Theorem

If $f(x)$ is a real function defined over $(-\infty, +\infty)$, and the integral $\int_{-\infty}^{+\infty} f(x)\,dx$ is absolutely convergent, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{+\infty} \cos k(\xi - x) f(\xi)\,d\xi$$

Proof

Since the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent, its Fourier transform and the inverse both exist. Therefore

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\,dk$$

Splitting the infinite integral in parts and using the fact that $f(x)$ is real, we have

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_{-\infty}^{0} e^{-\iota kx} F(k)\, dk\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_{\infty}^{0} e^{\iota k'x} F(-k')(-dk'), \quad k' = -k \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-\iota kx} F(k)\, dk + \int_0^{\infty} e^{\iota k'x} F(-k')\, dk'\right]
\end{aligned}
$$

Since $f(x)$ is real, by conjugation property, we must have $F(-k) = \bar{F}(k)$. Therefore we can write

$$
f(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[e^{-\iota kx} F(k) + e^{\iota kx}\bar{F}(k)\right] dk
$$

Now

$$
F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota kx'} f(x')\, dx'
$$

Therefore

$$
e^{-\iota kx} F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\iota k(x-x')} f(x')\, dx'
$$

and taking complex conjugate of both sides

$$
e^{\iota kx}\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\iota k(x-x')} f(x')\, dx'
$$

Adding the last two equations, we have

$$
\begin{aligned}
e^{-\iota kx} F(k) + e^{\iota kx}\bar{F}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x')\left[e^{\iota k(x-x')} + e^{-\iota k(x-x')}\right] dx' \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\, 2\cos k(x-x')\, dx'
\end{aligned}
$$

Substituting in the equation for $f(x)$, we obtain

$$
f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(x')\cos k(x-x')\, dx'\, dk
$$

7.8 Fourier Sine and Cosine Transforms

If a function $f(x)$ is defined over the interval $[0, \infty)$, then we can define its Fourier sine transform $F_s(k)$ or $\mathcal{F}_s\{f(x)\}$ as

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx \tag{7.8.1}$$

Similarly the Fourier cosine transform is defined as

$x \to k$

$$F_c(k) \equiv \mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx \tag{7.8.2}$$

The inverse relations to (7.8.1) and (7.8.2) are given by

$\boxed{t \to s}$

$$f(x) \equiv \mathcal{F}_s^{-1}\{F_s(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \, dk \tag{7.8.3}$$

and

$$f(x) \equiv \mathcal{F}_c^{-1}\{F_c(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx \, dk \tag{7.8.4}$$

7.8.1 Justification for the Definitions

The above definitions follow directly from the definition of complex or exponential Fourier transform. Let a real function $f(x)$ be defined over $[0, \infty)$. Its even extension $f_e(x)$ over the whole real line can be defined as

$$f_e(x) = \begin{cases} f(x), & \text{for } 0 \leq x < \infty \\ f(-x), & \text{for } -\infty < x < 0 \end{cases}$$

Now the Fourier transform of $f_e(x)$ is given by

$$\begin{aligned} \mathcal{F}\{f_e(x)\} \equiv F_e(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx} f_e(x) \, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_0^{+\infty} e^{ikx} f(x) + \int_{-\infty}^{0} e^{ikx} f(-x) \right] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left[e^{ikx} + e^{-ikx} \right] f(x) \, dx \end{aligned}$$

or

$$F_e(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx \, f(x) \, dx$$

Similarly if we define the odd extension of $f(x)$ over the whole real line as

$$f_o(x) = \begin{cases} f(\ x), & \text{for } 0 \le x < \infty \\ -f(-x), & \text{for } \infty < x < 0 \end{cases}$$

and perform similar calculations, we at the definition of the Fourier sine transform given above.

7.9 Fourier Sine and Cosine Transforms of Derivatives

To calculate Fourier sine and cosine transforms of first order derivative, we assume that (i) $f(x)$ is real and (ii) $|f(x)| \to 0$ as $x \to \infty$. Then

$$\begin{aligned}
\mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos kx\, dx \\
&= \sqrt{\frac{2}{\pi}} \left[\cos kx\, f(x)]_0^\infty + k \int_0^\infty f(x) \sin kx\, dx \right] \\
&= \sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} f(0) + k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx\, dx
\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + kF_s(k)} \tag{7.9.1}$$

Similarly

$$\begin{aligned}
\mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin kx\, dx \\
&= \sqrt{\frac{2}{\pi}} f(x) \sin kx]_0^\infty - k \int_0^\infty f(x) \cos kx\, dx \\
&= -\sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} \times 0 - k\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx\, dx
\end{aligned}$$

or

$$\boxed{\mathcal{F}_s\{f'(x) = -k\, F_c(k)} \tag{7.9.2}$$

To calculate Fourier sine and cosine transforms of second order derivatives. we assume further that (iii) $|f'(x)| \to 0$, as $x \to \infty$, then

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[[\sin kx\, f'(x)]_0^\infty - k\int_0^\infty f'(x)\cos kx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - k\int_0^\infty f'(x)\cos kx\, dx\right]$$

$$= -k\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos kx\, dx$$

$$= -k\,\mathcal{F}_c\{f'(x)\}$$

$$= -k\left[-\sqrt{\frac{2}{\pi}}f(0) + kF_s(k)\right]$$

Therefore

$$\mathcal{F}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}k\, f(0) - k^2 F_s(k)$$

Z

Similarly

$$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[[f'(x)\cos kx]_0^\infty + k\int_0^\infty f'(x)\sin kx dx\right]$$

$$= 0 - \sqrt{\frac{2}{\pi}}f'(0) + k\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin kx\, dx$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + k\mathcal{F}_s\{f'(x)\}$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + k[-kF_c(k)]$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) - k^2F_c(k)$$

7.10 Illustrative Examples

Example 1

Calculate Fourier sine transform of the function $f(x) = e^{-x}\cos x$.

Solution

By definition

$$\mathcal{F}_s\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x \sin kx\,dx$$

Using the result $\sin A\cos B = (1/2)[\sin(A+B)+\sin(A-B)]$, we have

$$\sin kx \cos x = \frac{1}{2}[\sin(k+1)x + \sin(k-1)x]$$

Therefore

$$\mathcal{F}_s\{e^{-x}\cos x\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\{\sin(k+1)x + \sin(k-1)x\}\,dx$$

Now we use the formula

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

and obtain

$$\begin{aligned}\int_0^\infty e^{-x}\sin(k+1)x\,dx &= \frac{e^{-x}}{1+(k+1)^2}\times \\ &\times \{[-\sin(k+1)x-(k+1)\cos(k+1)x\}]_0^\infty \\ &= 0-\frac{1}{k^2+2k+2}(0-(k+1)) \\ &= \frac{k+1}{k^2+2k+2}\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty e^{-x}\sin(k-1)x\,dx &= \frac{e^{-x}}{1+(k-1)^2}\times \\ &\times \{-\sin(k-1)x-(k-1)\cos(k-1)\}]_0^\infty \\ &= 0-\frac{1}{k^2-2k+2}(0-(k-1)) \\ &= \frac{k-1}{k^2-2k+2}\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{F}_s\{e^{-x}\cos x\} &= \frac{1}{\sqrt{2\pi}}\left[\frac{k+1}{k^2+2k+2}+\frac{k-1}{k^2-2k+2}\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{2k^3}{k^4+4} = \frac{2}{\sqrt{\pi}}\frac{k^3}{k^4+4}\end{aligned}$$

Example 2

Calculate Fourier sine transform of the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi, \\ 0, & x > \pi \end{cases}$$

Solution

$$\begin{aligned}
\mathcal{F}_s\{f(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin kx \, dx + \sqrt{\frac{2}{\pi}} \int_\pi^\infty f(x) \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin kx \, dx + \sqrt{\frac{2}{\pi}} \int_\pi^\infty 0 \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin kx \, dx \\
&= -\frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\pi [\cos(k+1)x - \cos(k-1)x] \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\pi [-\cos(k+1)x + \cos(k-1)x] \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\sin(k+1)x}{k+1} + \frac{\sin(k-1)x}{k-1} \right\} \Big]_0^\pi \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{\sin(k+1)\pi}{k+1} + \frac{\sin(k-1)\pi}{k-1} \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} [-(k-1)\sin(k+1)\pi + (k+1)\sin(k-1)\pi] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} [(k-1)\sin k\pi - (k+1)\sin k\pi] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2-1} (-2\sin k\pi) \checkmark \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin k\pi}{1-k^2}
\end{aligned}$$

~~Sink k~~

$\sin(k\pi + \pi)$

$\sin k\pi \cos \pi$

Example 3

Show that

(a) $$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{2a^2k}{(a^2+k^2)^2}.$$

$$\text{(b)} \quad \mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2}$$

where $\mathcal{F}_s$ and $\mathcal{F}_c$ denote Fourier sine and cosine transforms.

Solution

We will use the results

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\frac{a}{a^2+k^2}$$

and

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\frac{k}{a^2+k^2}$$

(a)

$$\begin{aligned}
\mathcal{F}_s\{xe^{-ax}\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty x(e^{-ax}\sin kx)\,dx \\
&= \sqrt{\frac{2}{\pi}}\left[\left\{x\frac{e^{-ax}}{(a^2+k^2)}(-a\sin kx - k\cos kx)\right\}_0^\infty \right.\\
&\left. + \frac{1}{a^2+k^2}\int_0^\infty e^{-ax}(a\sin kx + k\cos kx)\,dx\right] \\
&= \sqrt{\frac{2}{\pi}}(0-0) + \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}\int_0^\infty e^{-ax}\sin kx \\
&+ \sqrt{\frac{2}{\pi}}\frac{k}{a^2+k^2}\int_0^\infty e^{-ax}\cos kx\,dx \\
&= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}\frac{e^{-ax}}{a^2+k^2}(-a\sin kx - k\cos kx)\Big]_0^\infty \\
&+ \sqrt{\frac{2}{\pi}}\frac{k}{a^2+k^2}\frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\Big]_0^\infty
\end{aligned}$$

On simplification

$$\begin{aligned}
\mathcal{F}_s\{xe^{-ax}\} &= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}\left(0+\frac{k}{a^2+k^2}\right) \\
&+ \sqrt{\frac{2}{\pi}}\frac{k}{a^2+k^2}\left(0+\frac{k}{a^2+k^2}\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{ak}{(a^2+k^2)^2} + \sqrt{\frac{2}{\pi}}\frac{ak}{(a^2+k^2)^2}
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \frac{2ak}{(a^2+k^2)^2}$$

(b)

Now we will prove that

$$\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-k^2}{(a^2+k^2)^2}, \quad a > 0$$

By definition

$$\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \cos kx \, dx$$

On integration by parts and using the results (1) and (2), we obtain

$$\begin{aligned}
\mathcal{F}_c\{xe^{-ax}\} &= \sqrt{\frac{2}{\pi}} \left[x \frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k \sin kx) \right]_0^\infty \\
&\quad - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{a^2+k^2} (-a\cos kx + k\sin kx)\, dx \\
&= \sqrt{\frac{2}{\pi}}(0-0) + \sqrt{\frac{2}{\pi}} \frac{a}{a^2+k^2} \int_0^\infty e^{-ax} \cos kx \, dx \\
&\quad - \sqrt{\frac{2}{\pi}} \frac{k}{a^2+k^2} \int_0^\infty e^{-ax} \sin kx \, dx \\
&= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+k^2} \left. \frac{e^{-ax}}{a^2+k^2\,(-a\cos kx + k\sin kx)} \right]_0^\infty \\
&\quad - \sqrt{\frac{2}{\pi}} \frac{k}{a^2+k^2} \frac{e^{-ax}}{a^2+k^2} (-a\sin kx - k\cos kx) \Big]_0^\infty \\
&= \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)^2}(0+a) + \sqrt{\frac{2}{\pi}} \frac{k}{(a^2+k^2)^2}(0-k) \\
&= \sqrt{\frac{2}{\pi}} \frac{a^2}{(a^2+k^2)^2} - \sqrt{\frac{2}{\pi}} \frac{k^2}{(a^2+k^2)^2} \\
&= \sqrt{\frac{2}{\pi}} \frac{a^2-k^2}{(a^2+k^2)^2}
\end{aligned}$$

$\int_0$

$\frac{a}{a^2+k^2}$

$\frac{k}{a^2+k^2}$

Example 4

Determine $\mathcal{F}_c\{x^{a-1}\}$ and $\mathcal{F}_s\{x^{a-1}\}$.

Solution

From definition

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \cos kx \, dx \tag{1}$$

and

$$\mathcal{F}_s\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \sin kx \, dx \tag{2}$$

To calculate the integrals on R.H.S. of equations (1) and (2), we define

$$f(z) = z^{\alpha-1} e^{-kz}, \quad 0 < \alpha < 1$$

If this is analytic in a contour C, then by Cauchy's theorem $\oint_C f(z)\, dz = 0$.

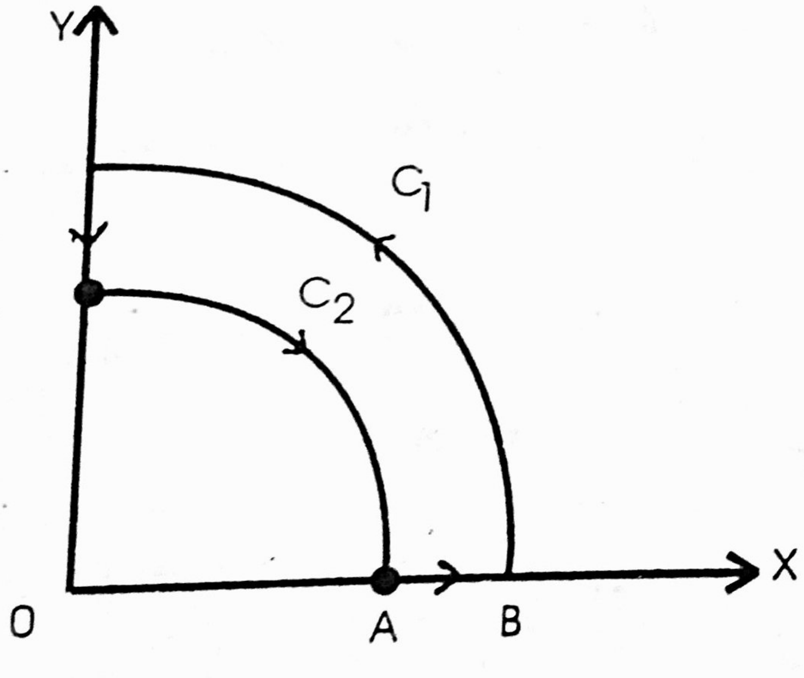

Figure 7.3:

We choose the contour C as shown in the figure. Therefore

$$\int_{C_1} f(z)\, z + \int_\epsilon^R x^{\alpha-1} e^{-kx}\, dx + \int_{C_2} f(z)\, dz$$
$$+ \int_R^\epsilon (\iota y)^{\alpha-1} e^{-k\iota y} (\iota\, dy) = 0 \tag{3}$$

In the limit $\epsilon \to 0$, $R \to \infty$ we can prove that $\int_{C_1} f(z)\, dz = 0$ and

$\int_{C_2} f(z)\,dz = 0$. Therefore from (3)

$$\int_0^\infty x^{\alpha-1}e^{-kx}\,dx = -\int_\infty^0 (\iota y)^{\alpha-1}e^{-k\iota y}(\iota dy) = \int_0^\infty \iota^\alpha y^{\alpha-1}e^{-\iota ky}\,dy$$

Now using the result

$$\iota^\alpha = \left(\cos\frac{\pi}{2} + \iota\sin\frac{\pi}{2}\right)^\alpha = e^{\iota\pi\alpha/2})$$

(4) can be written as

$$\int_0^\infty y^{\alpha-1}e^{-\iota ky}\,dy = e^{-\iota\pi\alpha/2}\int_0^\alpha x^{\alpha-1}e^{-kx}\,dx$$

Separating the real and imaginary parts, we obtain

$$\int_0^\infty y^{\alpha-1}\cos ky\,dy = \cos\frac{\pi\alpha}{2}\int_0^\infty x^{\alpha-1}\,e^{-kx}\,dx$$

and

$$\int_0^\infty y^{\alpha-1}\sin ky\,dy = \sin\frac{\pi\alpha}{2}\int_0^\alpha x^{\alpha-1}e^{-kx}\,dx$$

On making substitutions in (1) and (2)

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\cos(\pi\alpha/2)\int_0^\infty t^{\alpha-1}e^{-kt}\,dt$$

Putting $kt = t'$, we get

$$\begin{aligned}
\mathcal{F}_c\{x^{\alpha-1}\} &= \sqrt{\frac{2}{\pi}}\cos\frac{\pi\alpha}{2}\int_0^\infty \left(\frac{t'}{k}\right)^{\alpha-1} e^{-t'}\frac{dt'}{k} \\
&= \sqrt{\frac{2}{\pi}}\cos\frac{\pi\alpha}{2}\frac{1}{k^\alpha}\int_0^\infty t'^{\alpha-1}e^{-t'}\,dt' \\
&= \sqrt{\frac{2}{\pi}}\cos(\pi\alpha/2)\frac{\Gamma(\alpha)}{k^\alpha}, \qquad \text{(Q.E.D.)}
\end{aligned}$$

Similarly we can prove (2).

7.10.1 Exercises

1. Show that

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\Gamma(\alpha)\frac{\cos(\pi\alpha/2)}{k^\alpha}$$

2. Show that

$$\mathcal{F}_s\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\Gamma(\alpha)\frac{\sin(\pi\alpha/2)}{k^\alpha}$$

3. Find the (complex or exponential) Fourier transform of $f(x) = e^{-\lambda x^2}\cos\beta x$, $(\lambda > 0)$.

4. Find the (complex or exponential) Fourier transforms of $e^{-\lambda x^2}\cos\beta x$ and $e^{-\lambda x^2}\sin\beta x$.

[Hint: Use the modulation property stated in section 7.3 and obtain the results

$$\mathcal{F}\{e^{-\lambda x^2}\cos\beta x\} = \frac{1}{2}\frac{1}{\sqrt{2\lambda}}\left[\exp\left(-(k+\beta)^2/4\lambda\right) + \exp\left(-(k-\beta)^2/4\lambda\right)\right]$$

and

$$\mathcal{F}\{e^{-\lambda x^2}\sin\beta x\} = \frac{1}{2i}\frac{1}{\sqrt{2\lambda}}\left[\exp\left(-(k+\beta)^2/4\lambda\right) - \exp\left(-(k-\beta)^2/4\lambda\right)\right]$$

5. Find the (complex or exponential) Fourier transform of $F(x) = \cos\beta x/(a^4 + x^4)$.

6. Find the (complex or exponential) Fourier transform of

$$f(x) = \begin{cases} \cos k_0 x, & |x| < N\pi/k_0 \\ 0, & |x| > N\pi/k_0 \end{cases}$$

7. Calculate the following:

(a) $\mathcal{F}_s\{f''(x)\}$, (b) $\mathcal{F}_s\{f^{iv}(x)\}$, (c) $\mathcal{F}_c\{f^{iv}(x)\}$

8. Show that

(a) $$\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}, \quad (a>0)$$

(b) $$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{k}{a^2+k^2}, \quad (a>0)$$

(c) $$\mathcal{F}\{e^{-a|x|}\} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}, \quad (a>0)$$