where

$$R_0 = \text{residue of } e^{st} F(s) \text{ at } s = 0$$
$$= \frac{d}{ds}\left\{\frac{e^{st}s^2}{s^2(s+1)}\right\}\bigg|_{s=0}$$
$$= \left[\frac{te^{st}}{s+1} - \frac{e^{st}}{(s+1)^2}\right]\bigg|_{s=0} = t - 1$$

and

$$R_1 = \text{residue of } e^{st} F(s) \text{ at } s = -1$$
$$= e^{st}\frac{s+1}{(s+1)s^2}\bigg|_{s=-1} = \frac{e^{st}}{s^2}\bigg|_{s=-1} = e^{-t}$$

Hence

$$f(t) = R_0 + R_1 = t - 1 + e^{-t}$$

6.12 Application to Partial Differential Equations

Let the partial differential equation involve the unknown variable $u(x, t)$. Then we'll take the Laplace transform of $u(x, t)$ w.r.t. the variable t.

$$L\{u(x, t)\} = U(x, s)$$

For example

$$L\{e^{-at} \sin \pi x\} = \frac{\sin \pi x}{s + a}$$

and

$$L\{\sin(x + t)\} = L\{\sin x \cos t + \cos x \sin t\}$$
$$= L\{\sin x \cos t\} + L\{\cos x \sin t\}$$
$$= \frac{s \sin x}{s^2 + 1} + \frac{\cos x}{s^2 + 1}$$

Similarly

$$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\, dt$$
$$= \frac{\partial}{\partial x}\int_0^\infty e^{-st}u(x, t)\, dt$$
$$= \frac{\partial}{\partial x}L\{u(x, t)\} = \frac{\partial}{\partial x}U(x, s)$$

and

$$L\left\{\frac{\partial u}{\partial t}\right\} = s\,L\{u\} - u(x,\,t)]_{t=0}$$
$$= sU(x,\,s) - u(x,\,0)$$

where we have used the derivative rule.

6.12.1 Illustrative Examples

In this subsection we discuss some applications of the Laplace transform formalism to the solution of B.V.Ps. associated with partial differential equations.

Example 1

Use Laplace transform method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 \le t < \infty \tag{1}$$

$$u(0,\,t) = 1, \quad u(1,\,t) = 1, \quad t > 0 \tag{2}$$

$$u(x,\,0) = 1 + \sin \pi x, \quad 0 < x < 1 \tag{3}$$

Solution

It is a (one dimensional) heat problem describing conduction of heat through a rod of unit length, whose end-points are maintained at zero temperature, and whose initial temperature profile is prescribed.

We denote the Laplace transform of $u(x,\,t)$ by $U(x,\,s)$, i.e. $L\{u(x,\,t)\} = U(x,\,s)$. From (1)

$$\frac{\partial^2}{\partial x^2}U(x,\,s) = L\{\frac{\partial}{\partial t}u(x,\,t)\}$$

or

$$\frac{\partial^2}{\partial x^2}U(x,\,s) = sU(x,\,s) - u(x,\,0)$$
$$= sU(x,\,s) - 1 - \sin \pi x$$

where we have used (3). Simplifying further, we have

$$\frac{\partial^2}{\partial x^2}U(x,\,s) - sU(x,\,s) = -(1 + \sin \pi x)$$

which is a non-homogeneous linear second order differential equation, whose solution is given by

$$U(x,\ s) = U_c + U_p$$

where

$$U_c = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

and

$$\begin{aligned} U_p &= \frac{-1}{D^2 - s}(1 + \sin \pi x) \\ &= \frac{-1}{D^2 - s}(1) - \frac{1}{D^2 - s}\sin \pi x \\ &= \frac{-1}{D^2 - s}e^{0x} - \frac{1}{D^2 - s}\sin \pi x \\ &= \frac{1}{s} - \frac{1}{-\pi^2 - s}\sin \pi x \\ &= \frac{1}{s} + \frac{1}{\pi^2 + s}\sin \pi x \end{aligned}$$

920.

Therefore

$$\begin{aligned} U(x,\ s) &= U_c + U_p \\ &= c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{1}{\pi^2 + s}\sin \pi x \end{aligned} \qquad (4)$$

Conditions (2) when translated in terms of the variable s become

$$U(0,\ s) = \frac{1}{s}, \quad U(1,\ s) = \frac{1}{s} \qquad (5)$$

From (4) and (5)

$$c_1 + c_2 + \frac{1}{s} + 0 = \frac{1}{s} \Longrightarrow c_1 + c_2 = 0 \qquad (6)$$

and

$$c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} + 0 = \frac{1}{s}$$

which gives

$$c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \qquad (7)$$

From (6) and (7) $c_1 = c_2 = 0$. Therefore

$$U(x,\ s) = \frac{1}{s} + \frac{1}{\pi^2 x}$$

Hence the required solution of the given problem is

$$\begin{aligned} u(x, t) &= L^{-1}\{U(x, s)\} \\ &= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{\sin \pi x}{\pi^2 + s}\right\} \\ &= 1 + \sin \pi x \, e^{-\pi^2 t} \end{aligned}$$

Example 2

Solve the problem by Laplace transform method

$$u_{tt}(x, t) = a^2 u_{xx}(x, t), \quad (t > 0, \ x > 0) \tag{1}$$

$$u(x, 0) = u_t(x, 0) = 0 \tag{2}$$

$$u(0, t) = f(t), \quad \lim_{x \to \infty} u(x, t) = 0 \tag{3}$$

Solution

we define

$$U(x, s) = \int_0^\infty u(x, t) e^{-st} \, dt$$

Taking the Laplace transform of both sides of (1) w.r.t t and using the formulas

$$L\{u_t(x, t)\} = sU(x, s) - u(x, 0)$$

and

$$L\{u_{tt}(x, t)\} = s^2 U(x, s) - u(x, 0) - u_t(x, 0)$$

we have

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) = a^2 \frac{\partial^2 U(x, s)}{\partial x^2}$$

or using the initial conditions (2)

$$s^2 U(x, s) = a^2 \frac{\partial^2 U(x, s)}{\partial x^2}$$

or

$$\frac{\partial^2}{\partial x^2} U(x, s) - \frac{s^2}{a^2} U(x, s) = 0$$

whose solution is given by

$$U(x, s) = c_1 e^{sx/a} + c_2 e^{-sx/a} \tag{4}$$

where c_1, c_2 may depend on s, but are constant w.r.t. x. From (3)

$$U(0,\ s) = F(s) \text{ and } \lim_{x\to\infty} U(x,\ s) = 0 \tag{3'}$$

From (4) and (3') noting that $s > 0$, we must have $c_1 = 0$. Therefore

$$U(x,\ s) = c_2 e^{-xs/a}$$

This will satisfy (3') if $c_2 = F(s)$. Therefore

$$\therefore\ U(x,\ s) = F(s)e^{-x/as}$$

Finally the solution is given by

$$\begin{aligned} u(x,\ t) &= L^{-1}\{U(x,\ s)\} = L^{-1}\left\{e^{-xs/a}F(s)\right\} \\ &= H(t-x/a)\ f(t-x/a) \end{aligned}$$

where

$$H(t - x/a)\,f(t - x/a) = \begin{cases} f(t - x/a), & t > x/a \\ 0, & t < x/a \end{cases}$$

Example 3

Solve the problem using the Laplace transform formalism

$$u_{tt}(x,\ t) = a^2\ u_{xx}(x,\ t) - g \tag{1}$$

$$u(x,\ 0) = u_t(x,\ 0) = 0 \tag{2}$$

$$u(0,\ t) = 0, \quad \lim_{x\to\infty} u_x(x,\ t) = 0 \tag{3}$$

Solution

We define

$$U(x,\ s) = \int_0^\infty u(x,\ t)e^{-st}\,dt \tag{4}$$

Also

$$L\{u_t(x,\ t)\} = sU(x,\ s) - u(x,\ 0) \tag{5}$$

and

$$L\{u_{tt}(x,\ t)\} = s^2U(x,\ s) - su(x,\ 0) - u_t(x,\ 0) \tag{6}$$

Taking Laplace transform of both sides of (1), we have

$$L\{u_tt\} = a^2L\{u_{xx}(x,\ t)\} - L\{g\}$$

or making substitutions from (5), (6) and (7), we have

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) = a^2 \frac{d^2}{dx^2} U(x, s) - \frac{g}{s}$$

Again using the initial conditions (2)

$$s^2 U(x, s) = a^2 \frac{d^2}{dx^2} U(x, s) - \frac{g}{s}$$

which on rearrangement becomes

$$\frac{d^2}{dx^2} U(x, s) - \frac{s^2}{a^2} U(x, s) = \frac{g}{a^2 s} \tag{7}$$

Equation (8) is a linear nonhomogeneous second order DE. Therefore its solution can be written as

$$U(x, s) = U_c + U_p \tag{8}$$

For complimentary function U_c of (7), we consider the DE

$$\frac{d^2}{dx^2} U(x, s) - \frac{s^2}{a^2} U(x, s) = 0$$

whose auxiliary equation is

$$m^2 - s^2/a^2 = 0 \implies m = \pm \frac{s}{a}$$

Therefore complementary function is given by

$$U_c = c_1 e^{sx/a} + c_2 e^{-sx/a}$$

and the particular integral U_p of (8) is given by $(D = d/dx)$

$$U_p = \frac{1}{D^2 - s^2/a^2} \frac{g}{a^2 s}$$

$$= \frac{g}{a^2 s} \frac{1}{D^2 - s/a^2} e^{0x}$$

$$= \frac{g}{a^2 s} \frac{1}{(0 - s^2/a^2)} = -\frac{g}{s^3}$$

Hence

$$U(x, s) = U_c + U_p = c_1 e^{sx/a} + c_2 e^{-sx/a} - \frac{g}{s^3}$$

Next using the B.C. (3)

$$c_1 + c_2 - \frac{g}{s^3} = 0 \tag{9}$$

Before applying the B.C. (4), we calculate $U_x(x, s) = 0$.

$$U_x(x, s) = \frac{x}{a} c_1 e - \frac{x}{a} c_2 e^{-sx/a}$$

The B.C. $\lim_{x \to \infty} U_x(x, s) = 0$, gives $c_1 = 0$.

Therefore from (9), we have $c_2 = g/s^3$. Hence

$$U(x, s) = \frac{g}{s^3} e^{-sx/a} - g/s^3$$

Taking inverse Laplace transform of both sides we get

$$L^{-1}\{U(x, s)\} = L^{-1}\{g\frac{e^{-sx/a}}{s^3}\} - L^{-1}\{\frac{g}{s^3}\}$$

$$u(x, t) = gL^{-1}\left\{\frac{e^{-sx/a}}{s^3}\right\} - gL^{-1}\left\{\frac{1}{s^3}\right\} \tag{10}$$

We know

$$L^{-1}\{e^{-ks} F(s)\} = H(t-k)\, f(t-k)$$

In (10), let $F(s) = 1/s^3$, and $k = x/a$. Then

$$f(t) = L^{-1}\{F(s)\} = \frac{t^2}{2}.$$

Therefore

$$f(t-k) = \frac{1}{2}(t - x/a)^2 = (at - x)^2/2a^2$$

and

$$H(t - k) = H(t - x/a)$$

Therefore

$$I = L^{-1}\left\{\frac{e^{-ks}}{s^3}\right\} = \frac{1}{2}(t - x/a)^2\, H(t - x/a)$$

Therefore on making substitutions

$$u(x, t) = \frac{g}{2}(at - x)^2/a^2\, H(t - x/a) - \frac{gt^2}{2}.$$

Example 4

Solve the problem using Laplace transform method:

$$u_{xx} = u_{tt}, \quad 0 < x < 1, \quad t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(1, t) = 0 \tag{2}$$

$$u(x, 0) = \sin \pi x, \quad u_t(x, 0) = -\sin \pi x \tag{3}$$

Solution

We define

$$U(x, s) = L\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t)\, dt$$

Also

$$L\{u_{tt}\} = s^2 U(x, s) - su(x, 0) - u_t(x, 0) \tag{4}$$

Taking Laplace transform of both sides

$$L\{u_{tt}\} = L\{u_{xx}\}$$

or

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) = \frac{d^2}{dx^2} U(x, s)$$

Using the I.Cs. (3) and (4), we have

$$s^2 U(x, s) - s \sin \pi x + \sin \pi x = \frac{d^2}{dx^2} U(x, s)$$

or

$$\frac{d^2}{dx^2} U(x, s) - s^2 U(x, s) = -s \sin \pi x + \sin \pi x$$

which is a linear nonhomogeneous second order differential equation, whose general solution can be written as

$$U(x, s) = U_c + U_p$$

For U_c, $m^2 - s^2 = 0, i.e. m = \pm s$. Therefore

$$U_c = c_1 e^{sx} + c_2 e^{-sx}$$

And the particular integral is given by

$$\begin{aligned} U_p &= \frac{1}{D^2 - s^2}\{-s \sin \pi x + \sin \pi x\} \\ &= \frac{1}{D^2 - s}(-s \sin \pi x) + \frac{1}{D^2 - s}(\sin \pi x) \\ &= -\frac{s \sin \pi x}{(-\pi^2 - s^2)} + \frac{1}{(-\pi^2 - s^2)} \sin \pi x \\ &= \frac{s \sin \pi x}{\pi^2 + s^2} - \frac{1}{\pi^2 + s^2} \sin \pi x \\ &= \frac{(s-1) \sin \pi x}{\pi^2 + s^2} \end{aligned}$$

Therefore the solution is given by

$$\therefore U(x, s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{(s-1) \sin \pi x}{\pi^2 + s^2}. \tag{5}$$

We note that the B.Cs. (3) and (4) are equivalent to the conditions $U(0, s) = 0$, $U(1, s) = 0$. Applying these conditions to the solution (6), we have

$$c_1 + c_2 = 0 \text{ and } c_1 e^s + c_2 e^{-s} = 0$$

On solving these equations we obtain, $c_1 = 0$, $c_2 = 0$.

Therefore

$$U(x, s) = +\frac{s \sin \pi x}{\pi^2 + s^2} - \frac{\sin \pi x}{\pi^2 + s^2}$$

Taking the inverse Laplace transform of both sides, we get

$$\begin{aligned} u(x, t) &= L^{-1}\{U(x, s)\} = \sin \pi x \, L^{-1}\{\frac{s}{s^2 + \pi^2}\} \\ &- \frac{\sin \pi x}{\pi} L^{-1}\{\frac{\pi}{s^2 + \pi^2}\} \end{aligned}$$

or finally

$$\begin{aligned} u(x, t) &= \sin \pi x \cos \pi t - \frac{1}{\pi} \sin \pi x \sin \pi t \\ &= \sin \pi x\{-\frac{1}{\pi} \sin \pi t + \cos \pi t\} \end{aligned}$$

Example 5

A bar of length ℓ is initially at constant temperature u_0. The end $x = \ell$ is kept at the constant temperature u and the end $x = 0$ is insulated. Assuming that the surface of the bar is insulated, find the temperature at any point x of the bar at any time $t > 0$.

Solution

The problem consists of solving the heat equation

$$u_{xx} = \frac{1}{k}u_t, \quad 0 < x < \ell, \quad t > 0 \tag{1}$$

subject to the boundary and initial conditions

$$u(\ell, t) = u_1, \quad u_x(0, t) = 0, \quad u(x, 0) = u_0 \tag{2a, b, c}$$

with $0 \leq x \leq \ell, \quad t \geq 0$.

Taking the L.T. of both sides of (1), w.r.t. t and using the definition

$$U(x, s) = \int_0^\infty u(x, t)e^{-st}dt$$

we have

$$\frac{\partial^2}{\partial x^2}U(x, s) = \frac{1}{k}\left[sU(x, s) - u(x, 0)\right]$$

Using the initial condition in (2) we have

$$\frac{d^2U(x, s)}{dx^2} - \frac{s}{k}U(x, s) = -\frac{u_0}{k} \tag{3}$$

The solution of the non-homogeneous second order linear DE (3) is

$$U(x, s) = U_c + U_p$$

where

$$U_c = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x}$$

and

$$U_p = \frac{1}{D^2 - s/k}\left(-\frac{u_0}{k}\right), \quad \left(D \equiv \frac{d}{dx}\right)$$

$$= \left(\frac{-u_0}{k}\right)\frac{1}{D^2 - s/k}e^{0x} = \frac{u_0}{s}$$

Therefore

$$U(x,s) = c_1' e^{\sqrt{s/k}x} + c_2' e^{-\sqrt{s/k}x} + \frac{u_0}{s}$$

which can also be written in terms of hyperbolic functions as

$$U(x,s) = c_1 \cosh\sqrt{\frac{s}{k}}x + c_2 \sinh\sqrt{\frac{s}{k}}x + \frac{u_0}{s} \tag{4}$$

Now the B.Cs in (2) can also be written as

$$U(\ell,s) = \frac{u_1}{s}, \quad \frac{dU}{dx}(0,s) = 0 \tag{5}$$

From (4):

$$\frac{dU}{dx}(x,s) = \sqrt{\frac{s}{k}}\left((c_1 \sinh\sqrt{\frac{s}{k}}x + c_2 \cosh\sqrt{\frac{s}{k}}x\right)$$

The condition $dU(0,s)/ds = 0$ gives $c_2 = 0$. Therefore (4) can be written as

$$U(x,s) = c_1 \cosh\sqrt{\frac{s}{k}}x + \frac{u_0}{s} \tag{6}$$

The B.C. $U(\ell,s) = u_1/s$ applied to (6) gives

$$c_1 \cosh\left(\sqrt{\frac{s}{k}}\ell\right) + \frac{u_0}{s} = \frac{u_1}{s} \quad \Longrightarrow c_1 = \frac{u_1 - u_0}{s\cosh(\ell\sqrt{s}/\sqrt{k})}$$

Hence the solution (6) can be written as

$$U(x,s) = \frac{(u_1 - u_0)\cosh\sqrt{s/k}\,x}{s\cosh(\ell\sqrt{s/k})} + \frac{u_0}{s}$$

Taking the inverse Laplace transform, we have

$$\begin{aligned} u(x,t) &= (u_1 - u_0)L^{-1}\left\{\frac{\cos\sqrt{s/k}\,x}{s\cosh\sqrt{s/k}\,\ell}\right\} + u_0 L^{-1}\left\{\frac{1}{s}\right\} \\ &= (u_1 - u_0)I_1 + u_0 \end{aligned}$$

where

$$\begin{aligned} I_1 &= L^{-1}\{g(s)\}; \quad g(s) = \left\{\frac{\cos\sqrt{s/k}\,x}{s\cosh(\sqrt{s/k}\,\ell)}\right\} \\ &= \frac{1}{2\pi i}\int_{\gamma-i}^{\gamma+i} e^{st} g(s) \end{aligned}$$

The integral on the right equals the sum of residues of $g(s)e^{st}$ to the right of the line Re $s = \gamma$. Now $g(s)$ has a simple pole at $s = 0$ and at those values of s which satisfy

$$\cosh\left(\sqrt{\frac{s}{k}}\ell\right) \Longrightarrow \sqrt{\frac{s}{k}}\ell = \left(n+\frac{1}{2}\right)\pi\imath, \quad n = 0, \pm 1, \pm 2, \ldots$$

These give $s \equiv s_n = -(n+1/2)^2\pi^2 k/\ell^2$.

Now

$$\begin{aligned} R_0 &= \text{residue of } e^{st}g(s) \text{ at } s=0 \\ &= \lim_{s\to 0} s\,g(s)\,e^{st} = \lim e^{st}\frac{\cosh(\sqrt{s/k}\,x)}{\cosh(\sqrt{s/k}\,\ell)} = 1 \end{aligned}$$

and

$$\begin{aligned} R_n &= \text{residue of } e^{st}g(s) \text{ at } s = s_n \\ &= \lim_{s\to s_n}\frac{(s-s_n)e^{st}\cosh\sqrt{s/k}\,x}{s\cosh\sqrt{s/k}\,\ell} \\ &= e^{s_n t}\lim\frac{s-s_n}{\cos\sqrt{s/k}\ell}\cdot\lim_{s\to s_n}\frac{\cosh\sqrt{s/k}x}{s} \\ &= \exp\left[-(2n+1)^2\pi^2kt/4\ell^2\right]\frac{2\sqrt{s_n k}}{\ell\sinh(\sqrt{s_n/k}\ell)}\,\frac{\cosh\sqrt{s_n/k}x}{s_n} \end{aligned}$$

The value $\sqrt{s_n k} = (n+1/2)\pi\imath k/\ell$ implies that $\sqrt{s_n/k}\,\ell = (n+1/2)\pi\imath$, and

$\sinh\sqrt{s_n/k}\,\ell = \sinh(n+1/2)\pi\imath = \imath\sin(n+1/2)\pi, = \imath(-1)^n, \quad n = 0, 1, 2, \cdots$.

Therefore $s_n = -(n+1/2)^2\pi^2 k/\ell^2$, and

$$\begin{aligned} R_n &= \exp\left[-(2n+1)^2\pi^2\frac{kt}{4\ell^2}\right]\cdot\frac{(2n+1)\pi\imath k/\ell}{\ell\imath(-1)^n} \\ &\cdot\frac{4\ell^2}{-(2n+1)^2\pi^2k}\cosh\left[\frac{(2n+1)\pi^2 2\imath x}{2\ell}\right] \\ &= \exp\left[\frac{-(2n+1)^2\pi^2kt}{4\ell^2}\right]\cdot\frac{4(2n+1)\pi k}{(-1)^n(2n+1)^2\pi^2 k}\cos\frac{(2n+1)\pi x}{2\ell} \\ &= \exp\left[\frac{-(2n+1)^2\pi^2kt}{4\ell^2}\right]\frac{4}{(-1)^n\pi(2n+1)}\cos\frac{(2n+1)\pi x}{2\ell} \end{aligned}$$

where we have used the results $\sinh \imath x = \imath \sin x$ and $\cos \imath x = \cos x$. Therefore finally

$$
\begin{aligned}
u(x,t) &= (u_1 - u_0)\sum_{n=0}^{\infty} R_n(x,t) + u_0 \\
&= \frac{4}{\pi}(u_1 - u_0)\sum_{n=0}^{\infty} \exp\left[\frac{-2(2n+1)^2\pi^2 kt}{4\ell^2}\right] \times \\
&\times \frac{(-1)^n}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2\ell} + u_0
\end{aligned}
$$

6.12.2 Exercises

1. Solve the problem defined by the equations

$$u_{tt}(x,\ t) = a^2 u_{xx}(x,\ t) - g$$

$$u(x,\ 0) = u(t,\ 0) = 0,\quad u(x,\ t) = 0,\quad \lim_{x\to\infty} u_x(x,\ t) = 0$$

2. Solve the PDE $u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0$, subject to the boundary and initial conditions $\dot{u}(0,\ t) = 0,\ u(1,\ t) = 0,\ u(x,\ 0) = \sin \pi x$, and $(\partial u/\partial t)(x,\ 0) = -\sin \pi x$.

[**Ans.** $u(x,\ t) = \sin \pi x \cos \pi t - (1/\pi) \sin \pi t$].

3. A semi-infinite uniform conducting rod is initially at zero tempera ture. At time $t > 0$ a constant temperature $u_0 > 0$ is applied and maintained at the nearby end of the rod. Formulate and solve the problem, using Laplace transforms.

(**Hint:** $ku_{xx} = u_t,\ u(x,0) = 0,\ u(0,t)$ and $u(x,t)$ are finite as $x \to \infty$).

$$(\textbf{Ans}: u(x,t) = L^{-1}\left\{\frac{u_0}{s}e^{-\sqrt{s/k}\,x}\right\} = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

4. Find a formal solution of the problem $ku_{xx} = u_t,\ t > 0,\ 0 < x < \infty$.

$u(0,t) = f(t),\ u(x,0) = 0,\ u(x,t) \to 0$ as $x \to \infty$

(**Ans:** $U(x,s) = F(s)e^{-\sqrt{s/k}\,x},\ u(x,t) = L^{-1}\left\{F(s)e^{-\sqrt{s/k}\,x}\right\}$).

****