where $c_1 = y'(0)$ and $c_2 = y(0)$ are constants. Therefore

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} \\ &= L^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + L^{-1}\left\{\frac{c_2 s}{s^2+k^2}\right\} \\ &+ L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2\cos kt + L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2\cos kt + \frac{1}{k}\sin kt \star f(t) \end{aligned}$$

where we have used the convolution theorem. Finally we have

$$y(t) = \frac{c_1}{k}\sin kt + c_2(\cos kt) + \frac{1}{k}\int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

Example 4

Solve the I.V.P.

$$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

Taking the Laplace transform of both the sides

$$L\{y''\} + L\{ty'(t)\} - L\{y(t)\} = 0$$

or

$$s^2Y(s) - sy(0) - y'(0) + (-\frac{d}{ds})\, L\{y'(t)\} - Y(s) = 0$$

or on substituting for the initial values, we have

$$s^2Y(s) - 1 - \frac{d}{ds}\{sY(s) - y(0)\} - Y(s) = 0$$

or

$$s^2Y(s) - 1 - \{sY'(s) + Y(s)\} - Y(s) = 0$$

or

$$(s^2-1)\,Y(s) - sY'(s) - Y(s) = 1$$

or

$$-s\,Y'(s) + (s^2-2)\,Y(s) = 1$$

or

$$Y'(s) + \frac{2-s^2}{s} Y(s) = \frac{-1}{s} \tag{1}$$

which is of the form $Y' + p(x)Y = q(x)$, (i.e. it is first order linear nonhomogeneous DE). Its integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp \int \left(\frac{2}{s} - s\right) ds = \exp(2\ln s - s^2/2) = s^2 \exp(-s^2/2)$$

Therefore general solution of (1) is given by

$$\begin{aligned} Y(s)s^2 e^{-s^2/2} &= \int \left(-\frac{1}{s}\right) s^2 e^{-s^2/2} ds + \text{constant} \\ &= \int \{-s\, e^{-s^2/2}\} ds + \text{constant} \\ &= e^{-s^2/2} + c \end{aligned}$$

where c is a constant. Therefore

$$Y(s) = \frac{1}{s^2} + c e^{s^2/2}$$

Now taking limit as $s \to \infty$, we have $c = 0$. Therefore $Y(s) = 1/s^2$.

Taking inverse Laplace transform, we obtain $y(t) = t$, as the required solution.

Example 5

Use Laplace transform to solve the problem

$$u'' - au = f(t), \quad u(0) = u_0, \quad u'(0) = u_1$$

Solution

Applying the Laplace transform operator on both sides, we get

$$L\{u''\} - aL\{u(t)\} = L\{f(t)\}$$

or

$$s^2 U(s) - su(0) - u'(0) - aU(s) = F(s)$$

or using the initial conditions

$$(s^2 - a)U(s) - su_0 - u_1 - aU(s) = F(s)$$

or

$$(s^2 - a)U(s) = F(s) + u_0 s + u_1$$

or

$$U(s) = \frac{F(s)}{s^2 - a} + \frac{u_1}{s^2 - a} + \frac{u_0 s}{s^2 - a}$$

Taking the inverse Laplace transform of both sides, we have

$$\begin{aligned} u(t) &= L^{-1}\{U(s)\} = L^{-1}\left\{\frac{F(s)}{s^2 - a}\right\} \\ &+ L^{-1}\left\{\frac{u_1}{s^2 - a}\right\} + L^{-1}\left\{\frac{u_0 s}{s^2 - a}\right\} \\ &= \frac{1}{\sqrt{a}} L^{-1}\left\{\frac{\sqrt{a}}{s^2 - a} F(s)\right\} + \frac{u_1}{\sqrt{a}} \sinh \sqrt{a}\, t + u_0 \cosh \sqrt{a}\, t \\ &= \frac{1}{\sqrt{a}} \sinh \sqrt{a}\, t \star f(t) + \frac{u_1}{\sqrt{a}} \sinh \sqrt{a}\, t + u_0 \cosh \sqrt{a}\, t \end{aligned}$$

where we have used the convolution theorem for the first term and the fact that $L^{-1}\{F(s)\} = f(t)$.

6.11 The Laplace Inversion Integral/Fourier-Mellin Integral

In this section we discuss a general method for calculating the inverse Laplace transform of a given function. The relevant theorem is given below.

Theorem

If $f(t)$ is the inverse Laplace transform of $F(s)$, and all the singularities of $F(s)$ in the complex s plane, ($s = x + \iota y$), lie to the left of the line $x = \gamma$, then

$$f(t) = \frac{1}{2\pi\iota} \lim_{\gamma\to\infty} \int_{\gamma-\iota\infty}^{\gamma+\iota\infty} e^{st} F(s)\, ds$$

Proof

Draw the line $x = \gamma$ in the s-plane and mark the points $A = (\gamma,\ R)$ and $B(\gamma,\ -R)$ on this line and draw a semicircle S of radius R to the right of the line $x = \gamma$. Let $C = \overline{AB} \cup S$ be the closed contour consisting

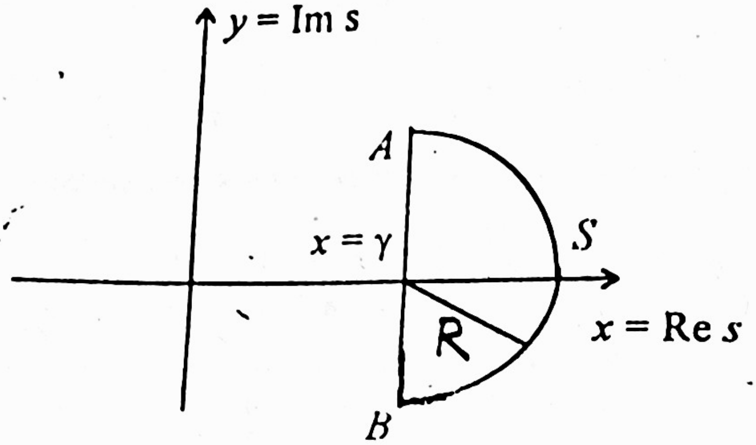

Figure 6.3:

of the line segment $\overline{AB}$ and S, as shown in figure 6.2. The function $F(z)$ is analytic on and within the contour C. If s is any point inside C then by Cauchy's integral theorem,

$$F(s) = \frac{-1}{2\pi\iota} \oint_C \frac{F(z)}{z-s}\, dz \tag{6.11.1}$$

where

$$F(z) = \int_0^{\infty} e^{-zt}\, f(t)\, dt \tag{6.11.2}$$

Substituting for $F(z)$ from (6.11.2) in (6.11.1), we have

$$F(s) = \frac{-1}{(2\pi\iota)} \oint_C \frac{1}{z-s} \int_0^{\infty} e^{-zt}\, f(t)\, dt\, dz$$

Interchanging the order of intergration, we have

$$F(s) = \frac{-1}{2\pi\iota} \int_0^{\infty} f(t) \left[\int_S \frac{e^{-zt}}{z-s}\, dz + \int_A^B \frac{e^{-zt}}{z-s}\, dz \right] dt \tag{6.11.3}$$

By Jordan's lemma, the integrand of the first integral on R.H.S. $\to 0$ as $R \to \infty$. Therefore this integral approaches 0. Hence as $R \to \infty$

$$\int_A^B \frac{e^{-zt}}{z-s}\, dz = \lim_{R\to\infty} \int_{\gamma+\iota R}^{\gamma-\iota R} \frac{e^{-zt}}{(z-s)}\, dz$$

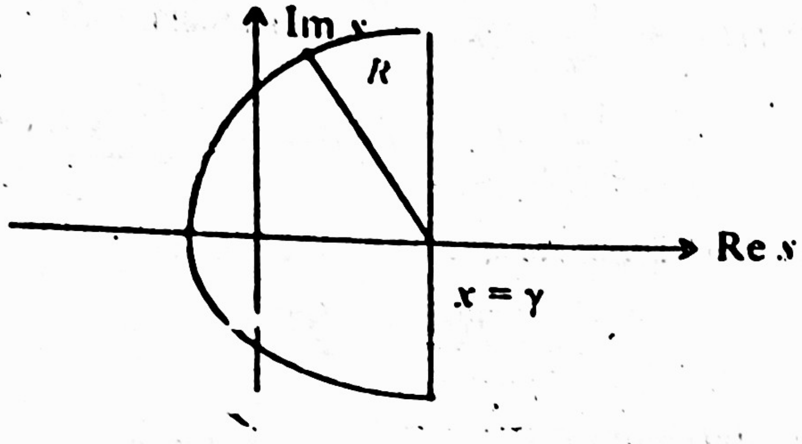

Figure 6.4:

Therefore from (6.11.3)

$$F(s) = \frac{-1}{2\pi\iota}\int_0^\infty f(t)\int_{\gamma-\iota\infty}^{\gamma+\iota\infty}\frac{e^{-zt}}{s-z}\,dz\,dt$$

Again changing the order of integration

$$\begin{aligned} F(s) &= \frac{1}{2\pi\iota}\int_{\gamma-\iota\infty}^{\gamma+\iota\infty}\left\{\int_0^\infty e^{-zt} f(t)\,dt\right\}\frac{1}{s-z}\,dz \\ &= \frac{1}{2\pi\iota}\int_{\gamma-\iota\infty}^{\gamma+\iota\infty}\frac{F(z)}{s-z}\,dz \end{aligned}$$

Applying L^{-1} to both sides

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi\iota}\int_{\gamma-\iota\infty}^{\gamma+\iota\infty} F(z)\,L^{-1}\{\frac{1}{s-z}\}\,dz$$

or

$$f(t) = \frac{1}{2\pi\iota}\int_{\gamma-\iota\infty}^{\gamma+\iota\infty} F(z)\,e^{zt}\,dz$$

or replacing z by s, we obtain

$$f(t) = \frac{1}{2\pi\iota}\int_{\gamma-\iota\infty}^{\gamma+\iota\infty} e^{st}\,F(s)\,ds$$

Suppose $F(s)$ has only poles to the left of the line $x = \text{Re}\, s = \gamma$. Then we can enclose all these poles in a contour C on the left of $x = \gamma$. Then

$$f(t) = \frac{1}{2\pi\iota} \lim_{R\to\infty} \int_{\gamma-\iota R}^{\gamma+\iota R} e^{st} F(s)\, ds$$
$$= \frac{1}{2\pi\iota} \oint_C e^{st} F(s)\, ds = \frac{1}{2\pi\iota} \sum_j (2\pi\iota R_j)$$

where we have used the residue theorem. Hence

$$f(t) = \sum_j R_j$$

where R_j = residue of $e^{st} F(s)$ at the pole $s = s_j$.

6.11.1 Illustrative examples

Example 1

Use the Laplace inversion integral to evaluate

$$L^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$$

Solution

Here

$$F(s) = \frac{s^3+2s^2+1}{s^2(s^2+1)}$$

The function $e^{st}\ F(s)$ has a double pole at $s = 0$ and it has simple poles at $s = \pm\iota$ Therefore by the Laplace inversion integral theorem $f(t) = R_0 + R_1 + R_2$, where

$$R_0 = \text{residue of the function } F(s)\, e^{st} \text{ at } s = 0$$
$$= \frac{d}{ds}\{s^2 e^{st} F(s)\}\Big]_{s=0}$$
$$= \frac{d}{ds}\left\{e^{st}\frac{s^3+2s^2+1}{s^2+1}\right\}\Big]_{s=0}$$
$$= \left[\frac{te^{st}(s^3+2s^2+1)}{s^2+1} + \frac{e^{st}(3s^2+4s)}{(s^2+1)}\right.$$
$$\left. + \frac{e^{st}\,(s^3+2s^2+1)\,(-2s)}{(s^2+1)^2}\right]\Bigg|_{s=0}$$
$$= t + 0 + 0 = t$$