we can enclose all these poles in a contour C on the left of $x = \gamma$. The

$$\begin{aligned} f(t) &= \frac{1}{2\pi\iota} \lim_{R\to\infty} \int_{\gamma-\iota R}^{\gamma+\iota R} e^{st} F(s)\, ds \\ &= \frac{1}{2\pi\iota} \oint_C e^{st} F(s)\, ds = \frac{1}{2\pi\iota} \sum_j (2\pi\iota R_j) \end{aligned}$$

where we have used the residue theorem. Hence

$$f(t) = \sum_j R_j$$

where R_j = residue of $e^{st} F(s)$ at the pole $s = s_j$.

6.11.1 Illustrative examples

Example 1

Use the Laplace inversion integral to evaluate

$$L^{-1}\left\{\frac{s^3 + 2s^2 + 1}{s^2(s^2 + 1)}\right\}$$

Solution

Here

$$F(s) = \frac{s^3 + 2s^2 + 1}{s^2(s^2 + 1)}$$

The function $e^{st} F(s)$ has a double pole at $s = 0$ and it has simple poles at $s = \pm\iota$ Therefore by the Laplace inversion integral theorem $f(t) = R_0 + R_1 + R_2$, where

$$\begin{aligned} R_0 &= \text{residue of the function } F(s)\, e^{st} \text{ at } s = 0 \\ &= \frac{d}{ds}\{s^2 e^{st} F(s)\}\Big|_{s=0} \\ &= \frac{d}{ds}\left\{e^{st}\frac{s^3 + 2s^2 + 1}{s^2 + 1}\right\}\Big|_{s=0} \\ &= \left[\frac{te^{st}(s^3 + 2s^2 + 1)}{s^2 + 1} + \frac{e^{st}(3s^2 + 4s)}{(s^2 + 1)}\right. \\ &\quad \left. + \frac{e^{st}(s^3 + 2s^2 + 1)(-2s)}{(s^2 + 1)^2}\right]\Bigg|_{s=0} \\ &= t + 0 + 0 = t \end{aligned}$$

and

$$\begin{aligned} R_1 &= \text{residue of the function } F(s)\,e^{st} \text{ at } s = \iota \\ &= \lim_{s\to\iota} e^{st}(s-\iota)\,F(s) \\ &= e^{st}\frac{(s^3+2s^2+1)}{s^2(s+\iota)}\Bigg|_{s=\iota} \\ &= e^{\iota t}\frac{(-\iota-2+1)}{(-2\iota)} \\ &= e^{it}\frac{(-\iota-1)}{(-2\iota)} = \frac{1}{2}e^{\iota t}(1-\iota) \end{aligned}$$

$$R_2 = \text{residue of the function at } s = -\iota = \frac{+1}{2}e^{-\iota t}(1+\iota)$$

and

$$\begin{aligned} f(t) &= t + \frac{1}{2}e^{\iota t}(1-\iota) + \frac{1}{2}e^{-\iota t}(1+\iota) \\ &= t + \frac{1}{2}[(\cos t + \iota \sin t)(1-\iota)] \\ &+ \frac{1}{2}[(\cos t - \iota \sin t)(1+\iota)] \\ &= t + \frac{1}{2}[2\cos t + 2\sin t + \iota \cos t] \\ &+ \frac{1}{2}[-\iota \cos t + \iota \sin t - \iota \sin t] \\ &= t + \frac{2}{2}\{\cos t + \sin t\} \\ &= t + \cos t + \sin t \end{aligned}$$

Example 2

Use the inversion integral (or residue) method to calculate the Laplace transform of the following:

$$\text{(a) } F(s) = \frac{2s+1}{s(s^2+1)}, \quad \text{(b) } F(s) = \frac{1}{s^2(s+1)}$$

Solution

(a) $F(s) = (2s+1)/[s(s^2+1)]$ and the function $e^{st}F(s) = e^{st}(2s+1)/[s(s^2+1)]$ has simple poles at $s=0$ and $s = \pm\iota$. Therefore

$$f(t) = L^{-1}\{F(s)\} = R_0 + R_1 + R_2$$

where

$$\begin{aligned} R_0 &= \text{residue of } e^{st} F(s) \text{ at } s = 0 \\ &= \lim_{s\to 0} e^{st} \frac{s(2s+1)}{s(s^2+1)} = 1\frac{(0+1)}{(0+1)} = 1 \end{aligned}$$

$$\begin{aligned} R_1 &= \text{residue of } e^{st} F(s) \text{ at } s = +\iota \\ &= \lim_{s\to \iota} e^{st} \frac{(s-\iota)(2s+1)}{(s-\iota)(s+\iota)s} \\ &= e^{\iota t}\frac{(2\iota+1)}{(-2)} = -\frac{1}{2}e^{\iota t}(2\iota+1) \end{aligned}$$

and

$$\begin{aligned} R_2 &= \text{residue of } e^{st} f(s) \text{ at } s = -\iota \\ &= \lim_{s\to -\iota} e^{st} \frac{(s+\iota)(2s+1)}{(s+\iota)(s+\iota)s} \\ &= e^{-\iota t}\frac{-2\iota+1}{-2} \\ &= \frac{-1}{2}e^{-\iota t}(-2\iota+1) \end{aligned}$$

Hence

$$\begin{aligned} f(t) &= R_0 + R_1 + R_2 \\ &= 1+\frac{1}{2}e^{\iota t}(2\iota+1)+(\frac{-1}{2})e^{\iota t}(-2\iota+1) \\ &= -\frac{1}{2}\{(\cos t + \sin t)(2\iota+1) + (\cos t - \sin t)(-2\iota+1)\} \\ &= 1-\frac{1}{2}\{2\iota\cos t - 2\sin t + \cos t + \iota \sin t \\ &- 2\iota\cos t - 2\sin t + \cos t - \iota\sin t\} \\ & 1-\frac{1}{2}\{-4\sin t + 2\cos t\} \\ &= 1 + 2\sin t - \cos t \end{aligned}$$

(b) $F(s) = 1/[s^2(s+1)]$. The function $e^{st}F(s) = e^{st}/[s^2(s+1)]$ has pole of order 2 at $s = 0$ and a simple pole at $s = -1$. Therefore

$$f(t) = L^{-1}\{F(s)\} = R_0 + R_1$$

where

$$
\begin{aligned}
R_0 &= \text{residue of } e^{st}\ F(s) \text{ at } s = 0 \\
&= \frac{d}{ds}\left\{\frac{e^{st}\, s^2}{s^2(s+1)}\right\}\Bigg|_{s=0} \\
&= \left[\frac{te^{st}}{s+1} - \frac{e^{st}}{(s+1)^2}\right]\Bigg|_{s=0} = t - 1
\end{aligned}
$$

and

$$
\begin{aligned}
R_1 &= \text{residue of } e^{st}\ F(s) \text{ at } s = -1 \\
&= e^{st}\,\frac{s+1}{(s+1)s^2}\Bigg|_{s=-1} = \frac{e^{st}}{s^2}\Bigg|_{s=-1} = e^{-t}
\end{aligned}
$$

Hence

$$
f(t) = R_0 + R_1 = t - 1 + e^{-t}
$$