6.5 The Inverse Laplace Transform

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is called the *inverse Laplace transform* of $F(s)$. The inverse Laplace transform of $F(s)$ is denoted by $L^{-1}\{F(s)\} = f(t)$.

In calculating inverse Laplace transforms of functions, we make use of our knowledge of Laplace transforms of simple functions, and develop special methods. The following table summarizes the results on Laplace transforms.

Table of Laplace Transforms

$f(t)$	$F(s)$
1	$1/s, \quad s > 0$
t^n (n = positive integer)	$n!/s^{n+1}, \quad s > 0$
e^{kt}	$1/(s-k), \quad s > k$
$\sin kt$	$k/(s^2+k^2), \quad s > 0$
$\cos kt$	$s/(s^2+k^2), \quad s > 0$
$\sinh kt$	$k/(s^2-k^2), \quad s > \|k\|$
$\cosh kt$	$s/(s^2-k^2), \quad s > \|k\|$
$e^{kt} f(t)$	$F(s-k), \quad s > k$
$t^n e^{kt}$	$n!/(s-k)^{n+1}, \quad s > k$
$f^n(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{n-1}(0)$

6.5.1 Computation of Inverse Laplace Transforms

The first shifting theorem can be written in the form

$$L^{-1}\{F(s-k)\} = e^{kt} f(t) = e^{kt} L^{-1}\{F(s)\}$$

By using the above theorem we can calculate inverse Laplace transforms of some of the functions. This is illustrated with the following example.

The method of partial fractions may also be used in calculating inverse Laplace transform This is illustrated in example **2 below.

6.5.2 Applications to initial value problems

The Laplace transforms have wide applications in initial and boundary value problems associated with ordinary and partial DEs. Their simplest applications and direct applications are in the solution of initial-value problems consisting of an ODE subject to initial conditions. Such applications are illustrated in example 5. The results on Laplace transforms and inverse Laplace transforms discussed in the preceding sections will be used in these examples.

6.6 Illustrative Examples and Exercises

Example 1

(a) Calculate the Laplace transform of t^α, where α is any real number.

(b) Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$.

Solution

(a)

$$L\{t^\alpha\} = \int_0^\infty e^{-st}\, t^\alpha\, dt$$

Making the substitution $st = u$ on the right side, we have

$$\begin{aligned} L\{t^\alpha\} &= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \frac{du}{s} \\ &= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u}\, u^\alpha\, du \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \end{aligned}$$

where we used the definition of the gamma function $\Gamma(x) \equiv \int_0^\infty e^{-u} u^{x-1}\, dx$.

(b) This follows from (a) by noting that $\Gamma(1/2) \equiv \sqrt{\pi}$, $L\{t^{1/2}\} \equiv (1/2s)\sqrt{\pi/s}$, $L\{t^{-1/2}\} \equiv \sqrt{\pi/s}$

Example 2

Using the result $L\{t^\alpha\} = (\alpha/s)\ L\{t^{\alpha-1}\}$, $(s > 0,\ \alpha > -1)$, and the result $L\{t^{-1/2}\} = \sqrt{\pi/s}$, find $L\{t^{k/2}\}$, where k is an odd positive integer.

Solution

Since k is an odd positive integer, we can put $k = 2m + 1$, where m is a positive integer.

$$L\{t^{k/2}\} = L\{t^{m+1/2}\} = \frac{m + 1/2}{s} L\{t^{m-1/2}\}$$

Repeated application of the same result gives

$$L\{t^{k/2}\} = \frac{m + 1/2}{s} \frac{m - 1/2}{s} \frac{m - 3/2}{s} \cdots \frac{3/2}{s} \frac{1/2}{s} L\{t^{-1/2}\}$$

Using the result $L\{t^{-1/2}\} = \sqrt{\pi/s}$, we obtain

$$\begin{aligned} L\{t^{k/2}\} &= \frac{(2m+1)(2m-1)(2m-3)\cdots 3 \cdot 1}{(2s)^{m+1}} \sqrt{\frac{\pi}{s}} \\ &= \frac{k(k-2)(k-4)\cdots 3 \cdot 1}{2^{k/2+1/2}} \sqrt{\frac{\pi}{s^{k+2}}} \end{aligned}$$

Example 3

Find the inverse Laplace transforms of the following functions.

$$\text{(a)} \ \frac{1}{s^2 + 2s} \quad \text{(b)} \ \frac{s}{s^2 + 2s}$$

Solution

$$\text{(a)} \ \frac{1}{s^2 + 2s} = \frac{1}{s^2 + 2s + 1 - 1} = \frac{1}{(s+1)^2 - 1}$$

Hence $F(s + 1) = 1/[(s + 1)^2 - 1]$. Therefore with $k = -1$

$$L^{-1}\{F(s+1)\} = e^{kt} L^{-1}\{F(s)\} = e^{-t} L^{-1}\{\frac{1}{s^2 - 1}\} = e^{-t} \sinh t$$

where we have used the shifting theorem.

$$\text{(b)} \ \frac{s}{s^2 + 2s} = \frac{s}{(s+1)^2 - 1} = \frac{s + 1 - 1}{(s+1)^2 - 1}$$

Hence

$$F(s+1) = \frac{s+1}{(s+1)^2 - 1} - \frac{1}{(s+1)^2 - 1}$$

and therefore

$$\begin{aligned} L^{-1}\{F(s+1)\} &= e^{-t} L^{-1} \frac{s}{s^2 - 1} - e^{-t} L^{-1} \frac{1}{s^2 - 1} \\ &= e^{-t} \cosh t - e^{-t} \sinh t \end{aligned}$$

where we have used the shifting theorem.

Example 4

Calculate $L^{-1}\{(s+4)/(s^2+3s+2)\}$.

Solution

We have

$$\frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+1)(s+2} = \frac{A}{s+1}+\frac{B}{s+2}$$

which gives

$$s+4 = A(s+2)+B(s+1)$$

Putting $s+1=0$ and $s+2=0$ successively, we obtain $A=3$ and $B=-2$.

Therefore

$$L^{-1}\{\frac{s+4}{s^2+3s+2}\} = L^{-1}\left\{\frac{3}{s+1}\right\}+L^{-1}\left\{\frac{-2}{s+2}\right\} = 3e^{-t}L^{-1}\left\{\frac{1}{s}\right\}-2e^{-2t}L^{-1}\{\frac{1}{s}\} = 3e^{-t}-2e^{-2t}$$

where we have used the shifting theorem and the result $L^{-1}\{1/s\}=1$.

Example 5

Solve the I.V.Ps:

(a) $u'-2u=0, \quad u(0)=1$

(b) $u''+4u'+3u=0, \quad u(0)=1, \quad u'(0)=0$

Solution

(a) We take the Laplace transform of both sides of the differential equation $u'-2u=0$, and obtain

$$L\{u'\}-2L\{u\}=0$$

Using the formula for Laplace transforms of derivatives, we have

$$sU(s)-u(0)-2U(s)=0$$

Now using the initial condition $u(0) = 1$, we obtain

$$U(s)(s-2) = 1, \implies U(s) = \frac{1}{s-2}$$

wherefrom

$$u(t) = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

Hence $u(t) = e^{2t}$ is the solution of the given I.V.P.

(b) Taking Laplace transform of both sides of the DE, we have

$$L\{u''\} + 4L\{u'\} + 3L\{u\} = 0$$

Next using the formulas

$$L\{u''(t)\} = s^2 U(s) - su(0) - u'(0), \quad L\{u'(t)\} = s\,U(s) - u(0)$$

we have

$$s^2 U(s) - s\,u(0) - u'(0) + 4\{sU(s) - u(0)\} + 3U(s) = 0$$

or on using the given initial conditions

$$s^2 U(s) - s - 0 + 4\{sU(s) - 1\} + .3U(s) = 0$$

or

$$U(s)\left(s^2 + 4s + 3\right) - s - 4 = 0 \implies U(s) = \frac{s+4}{s^2+4s+3}$$

Hence on taking inverse Laplace transform, we obtain

$$u(t) = L^{-1}\left\{\frac{s+4}{s^2+4s+4}\right\} = L^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\}$$

Now

$$\frac{s+4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

On putting $s+1=0$ and $s+3=0$ one after the other, we obtain $A = 3/2$, $B = -1/2$. Therefore we have

$$\begin{aligned} L^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\} &= \frac{3}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+3}\right\} \\ &= \frac{3}{2}e^{-t}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}e^{-3t}L^{-1}\left\{\frac{1}{s}\right\} \\ &= \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{aligned}$$

6.6.1 Exercises

1. Compute Laplace transforms of the following functions

(a) $\sin^2 \omega t$, (b) $\cos^2 \omega t$, (c) $\sin(\omega t - \phi)$
(d) $\cos(\omega t - \phi)$, (e) $e^{2(t+1)}$

2. Find the Laplace transforms of the following functions.
(a) $\sin \omega t\, e^{-2t}$, (b) $\cos 3\omega t\, e^{4t}$, (c) $e^{3t}\, t^4$.

(Hints for solution
(2 a) $L\{e^{-2t} \sin \omega t\} = L\{\sin \omega t\}\,]_{s \to s+2} = \omega/(s^2 + 4s + 4 + \omega^2)$.
(2 c) $L\{e^{3t}\, t^4\} = L\{t^4\}]_{s \to s-3} = 4!/(s-3)^5$.

3. Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$ from definition.
(Hint: Use the substitutions $st = x$, where $x > 0$, and $x = y^2$, and the result $\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}$.)

Find the inverse Laplace transform of the following functions:

4. $s/(s^2 + 2as + b^2)$; 5. $1/(s^2 + 2s + 10)$.
6. $1/(s^2 - 4s + 8)$, 7. $s/(s^2 + 6s + 13)$.
8. $(2s + 3)/(s + 4)^3$, 9. $s^2/(s - 1)^4$.
10. $(2s - 3)/(s^2 - 4s + 8)$.

Use the method of partial fractions to calculate inverse Laplace transforms of the following functions:

11. $1/(s^2 - 4)$, 12. $1/(s^2 - 4)$.
13. $(s + 3)/[s(s^2 + 2)]$, 14. $4/[s(s + 1)]$.

15. Solve the I.V.Ps:
(a) $u' + 2u = 0$, $u(0) = 1$.
(b) $u'' + 9u = 0$, $u(0) = 0$, $u'(0) = 1$.

6.7 The Convolution Theorem and its Applications

6.7.1 The unit step function

We begin our discussion with defining *unit step function* which is also called *Heaviside step function.* We denote it by the symbol $H(t - t_0)$ and define it by the relations

$$H(t - t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases}$$

It is clear that the step function is discontinuous at $t = t_0$, (see fig. 6.1). Some important properties of the step function are illustrated by the following examples.

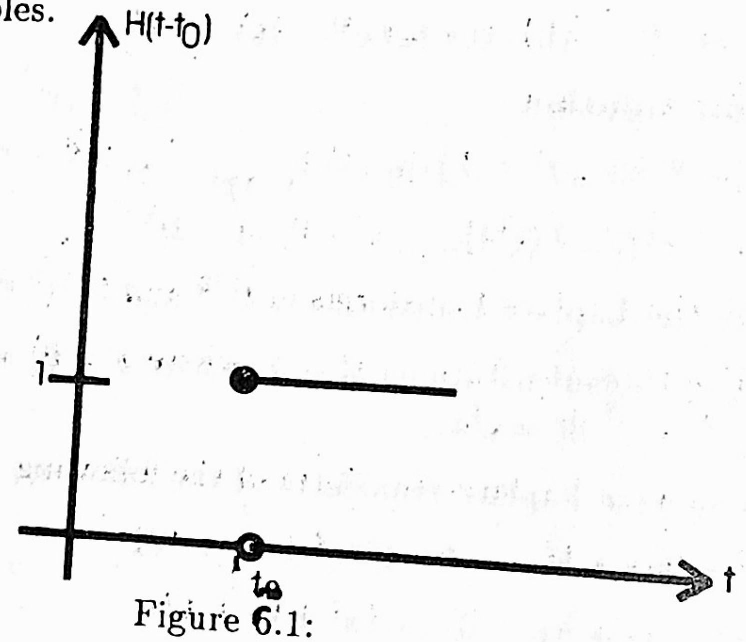

Figure 6.1:

Example 1

Sketch the graphs of the following functions:
(a) $H(t)\,t^2$, (b) $H(t-1)\,t^2$, (c) $H(t-2)\,t^3$.

Solution

(a) $f(t) = H(t)\,t^2 = \begin{cases} t^2, & t > 0 \\ 0, & t \leq 0 \end{cases}$ The graph is shown in fig. 6.2 (a).

(b) $f(t) = H(t-1)\,t^2 = \begin{cases} t^2, & t > 1 \\ 0, & t \leq 1 \end{cases}$ The graph is shown in fig. 6.2

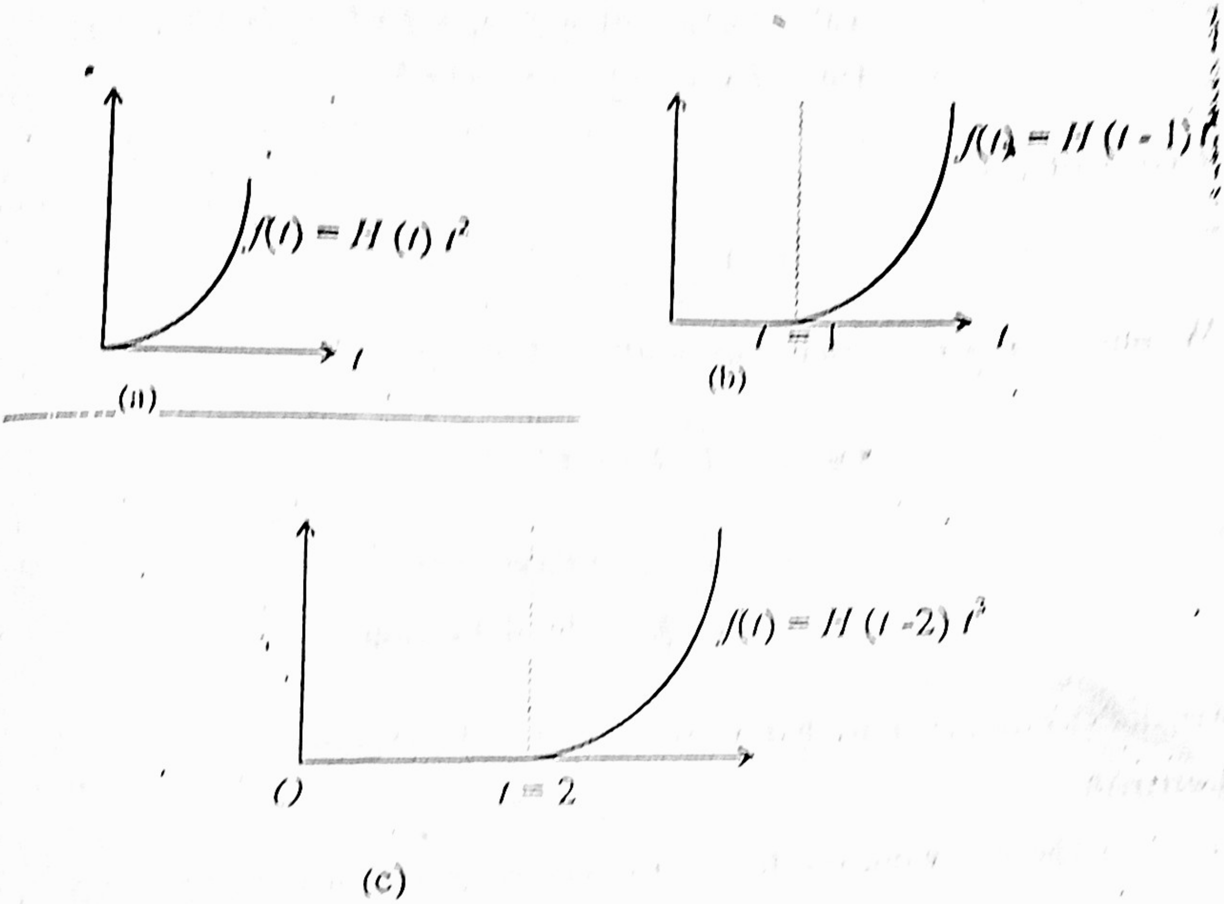

Figure 6.2:

(b).

(c) $f(t) = H(t-2)\,t^3 = \begin{cases} t^3, & t > 2 \\ 0, & t \leq 2 \end{cases}$. The graph is shown in fig. 6.2 (c).

6.7.2 The convolution theorem (or Faltung theorem)

If $f(t)$ and $g(t)$ are piecewise continuous functions over the interval $[0, \infty)$, then their *convolution* denoted by $f \star g$ is a function of t defined by the integral

$$f \star g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

From this definition it is easy to prove the following results.

(i) $f \star g = g \star f$

(ii) $f \star (g + h) = f \star g + f \star h$

(iii) $f \star (g \star h) = (f \star g) \star h$

Proof of (i)

$$f \star g \doteq \int_0^t f(\tau) g(t - \tau)\, d\tau$$

We put $t - \tau = t'$, so that $-d\tau = dt'$, and obtain

$$\begin{aligned} f \star g &= \int_t^0 f(t - t')\, g(t')\, (-dt') \\ &= \int_0^t f(t - t')\, g(t')\, dt' \\ &= g \star f, \quad \text{by definition} \end{aligned}$$

Results (ii) and (iii) are left as exercises for the student.

Lemma

We need the following result to prove the convolution theorem.

$$f \star g = \int_0^\infty H(t - \tau)\, f(\tau)\, g(t - \tau)\, d\tau$$

where H is the step function.

Using the defining property of the unit step function *viz.* $H(t-\tau) = 1$ when $t \geq \tau$ and $= 0$ when $t < \tau$, we have

$$\begin{aligned} f \star g &= \int_0^t f(\tau)\, g(t - \tau)\, d\tau \\ &= \int_0^t H(t - \tau)\, f(\tau)\, g(t - \tau)\, d\tau, \quad (t - \tau \geq 0) \\ &= \int_0^t H(t - \tau)\, f(\tau)\, g(t - \tau)\, d\tau \\ &+ \int_t^\infty H(t - \tau)\, f(\tau)\, g(t - \tau)\, d\tau \\ &= \int_0^\infty H(t - \tau)\, f(\tau)\, g(t - \tau)\, d\tau \end{aligned}$$

as required.