Chapter 6

LAPLACE TRANSFORM AND ITS APPLICATIONS

6.1 Integral Transforms

The concept of integral transformation is related to that of a linear transformation defined by an integral. To understand this particular linear transformation, we consider the set of functions of x over an interval $I : a \leq x \leq b$ which may be finite or infinite. Next we choose a fixed function $K(x, y)$ of variables (x, y). Then the integral transformation is defined by

$$T\{f(x)\} \equiv F(y) = \int_a^b f(x)\, K(x, y)\, dx$$

The function $K(x, y)$ is called the *kernel* of the transformation T. This concept has been very seminal and conducive to opening up new vistas in Modern Mathematics. In classical analysis certain special types of integral transformations such as Laplace, Fourier, Chebyshev have been extensively studied and used in solving various types of problems. Different transformations correspond to different forms for the kernel $K(x, y)$. The limits of the integral $[a, b]$ are also different in each case. The function $T\{f(x)\}$ obtained by means of such a transformation is called *intagral transform* of the given function $f(t)$.

6.2 Some Definitions and Basic Results

Let $f(t)$ be a continuous or sectionally continuous function of t defined over the interval $[0, \infty)$, then the *Laplace transform* of $f(t)$ is a function $F(s)$ of another variable s defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^{T} e^{-st} f(t)\, dt$$

It is to be noted that the existence of Laplace transform of a function depends on the existence of the defining integral. It is clear that every function may not possess its Laplace transform.

6.2.1 Notation

The Laplace transform of a function $f(t)$ is denoted by any one of the notations $F(s)$, $\bar{f}(s)$, $L\{f(t)\}$, $L[f(t)]$, $L\{f(t);\, s\}$. The functions $f(t)$ and $F(s)$ are referred to as it Laplace transform pair.

Theorem

The Laplace transformation operator L is a linear operator.

Proof

Let

$$f(t) = c_1 f_1(t) + c_2 f_2(t), \quad 0 \le t < \infty$$

Then

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\}dt \\ &= c_1 \int_0^{\infty} e^{-st} f_1(t) + c_2 \int_0^{\infty} e^{-st} f_2(t)dt \\ &= c_1 L\{f_1\} + c_2 L\{f_2\} \end{aligned}$$

which shows that the operator L is linear.

6.2.2 Existence of Laplace Transform

Some Definitions

A function $f(t)$ is said to be of *exponential order* c if

$$|f(t)| \le M e^{ct}, \quad \text{for } t > t_0$$

where M and t_0 are positive constants and c is also a constant.

A function which is piecewise continuous and of exponential order is said to be a function of *class A*.

Theorem

Sufficient condition for the existence of Laplace transform of a function is that it should be a function of class A. *i.e.* it should be piece-wise continuous and of exponential order.

Proof

Let $f(t)$ be piecewise continuous in the interval $[0, T]$ and be of exponential order c. Then it will be integrable over $[0, T]$ and moreover

$$|f(t)| \leq M\, e^{ct}, \quad \text{for } t > t_0$$

Therefore

$$|\, e^{-st} f(t)\,| \leq \; M e^{-(s-c)t}, \quad \text{for } t > t_0$$

Hence

$$\begin{aligned} |F(s)| &= |L\{f(t)\}| \leq \int_0^\infty M\, e^{-st} e^{ct}\, dt \\ &\leq M \left| \int_0^\infty e^{-(s-c)t}\, dt \right| \\ &\leq \frac{M}{|s-c|}, \quad \text{provided } s > c \end{aligned}$$

Thus it is clear that piecewise continuity and exponential order are sufficient conditions for a function to have a Laplace transform. But these are not necessary conditions. This can be seen from the fact that the function $t^{-\frac{1}{2}}$ has Laplace transform; yet it is not piecewise continuous in any interval $[0, T]$ where $T > 0$.

Corollary

From the above it follows that

$$\lim_{s \to \infty} F(s) = 0$$

This result is quite general and can be proved for the Laplace transform of any function, whether satisfying the conditions of the above theorem or not. Hence it follows from this result (corollary) that if $F(s)$ is any function of s such that its limit as $s \to \infty$ does not exist or is not zero, then it cannot be the Laplace transform of any function $f(t)$. Hence functions such as $F(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$ or $\ln s$, e^s, $\sin s$, $\cos s$ cannot be Laplace transforms of any functions. On the other hand a rational function is Laplace transform of some function if the degree of the numerator is less than that of the denominator.

6.3 Laplace Transforms of Some Functions and Basic Results

6.3.1 Laplace Transform of a constant

If $f(t) = k$, then

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} k\,dt \\ &= k \lim_{T\to\infty} \int_0^T e^{-st}\,dt \\ &= k \lim_{T\to\infty} \left.\frac{e^{-st}}{-s}\right]_0^T \\ &= \frac{k}{s} \lim_{T\to\infty} \left(1 - e^{-sT}\right) \\ &= k/s \end{aligned}$$

where it has been assumed that s or $\mathrm{Re}\, s > 0$; otherwise the limit will not exist. Hence we can write

$$L\{k\} = \frac{k}{s}, \quad s > 0$$

6.3.2 Laplace transform of a positive integral power of t

First we will calculate Laplace transforms of t and t^2 and then use these results to obtain $L\{t^n\}$.

(i) Let $f(t) = t$, then using Kronecker's rule for integration by parts

$$\begin{aligned} L\{t\} &= \int_0^\infty e^{-st} t\,dt \\ &= \left.\frac{te^{-st}}{-s} - 1\,\frac{e^{-st}}{-s^2}\right]_0^\infty \end{aligned}$$

$$= 0 - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

where we have used the result $\lim \left(t^n e^{-st}\right) = 0$, as $t \to \infty$ and for $\text{Re}\, s > 0$.

(ii) Let $f(t) = t^2$, then again using Kronecker's rule

$$\begin{aligned} L\{t^2\} &= \int_0^\infty e^{-st} t^2 dt \\ &= \left[\frac{t^2 e^{-st}}{-s} - 1\,(2t)\frac{e^{-st}}{-s^2} + \frac{e^{-st}}{(-s)^3}\right]_0^\infty \\ &= 0 + 0 + \frac{1}{s^3} = \frac{1}{s^3} \end{aligned}$$

(iii) Let $f(t) = t^n$, then from definition

$$\begin{aligned} L\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\ &= t^n \frac{e^{-st}}{-s}\Big]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt \\ &= 0 + \frac{n}{s} L\{s^{n-1}\} = \frac{n}{s} L\{s^{n-1}\} \end{aligned}$$

Again using the same result

$$\begin{aligned} L\{t^n\} &= \frac{n}{s}\frac{n-1}{s} L\{s^{n-2}\} \\ &= \frac{n(n-1)(n-2)\cdots 2\,1}{s^n} L\{1\} \\ &= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}} \end{aligned}$$

6.3.3 Laplace transforms of exponential and trigonometric functions

(i) Let $f(t) = e^{kt}$, then by definition

$$\begin{aligned} L\{e^{kt}\} &= \int_0^\infty e^{-st} e^{kt} \, dt \\ &= \int_0^\infty e^{-(s-k)\,t} \, dt \end{aligned}$$

$$= \int_0^\infty e^{-s't}\,dt \quad \text{where } s' = s - k$$

$$= \frac{1}{s'} = \frac{1}{s-k}$$

where it is assumed that Re $s > k$.

(ii) Let $f(t) = \sin kt$, where k is a constant. In calculating Laplace transforms for $\sin kt$ and $\cos kt$, we will use the following formulas

$$\int e^{at} \cos bt\,dt = \frac{e^{at}}{a^2+b^2}[a\cos bt + b\sin bt]$$

and

$$\int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2}[a\sin bt - b\cos bt]$$

By definition

$$L\{\sin k\,t\} = \int_0^\infty e^{-st} \sin kt\,dt$$

Using the above formula with $a = -s,\ b = k$, we have

$$L\{\sin kt\} = \left[\frac{e^{-st}}{s^2+k^2}(-s\sin kt - k\cos kt)\right]_0^\infty$$

$$= \frac{k}{s^2+k^2}$$

(iii) Let $f(t) = \cos kt$, then performing calculation similar to the above, with $a = -s,\ b = k$, we have

$$L\{\cos k\ t\} = \int_0^\infty e^{-st} \cos kt\,dt$$

$$= \left[\frac{e^{-st}}{s^2+k^2}(-s\cos kt + k\sin kt)\right]_0^\infty$$

$$= \frac{s}{s^2+k^2}$$

6.3.4 Laplace transforms of hyperbolic functions

These can be calculated using the definitions

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}$$

We obtain

$$L\{\sinh t\} = \frac{k}{s^2 - k^2} \text{ and } L\{\cosh t\} = \frac{s}{s^2 - k^2}$$

6.3.5 The first shifting theorem and the rule of scales

In this subsection we discuss two important results which are useful additions to our toolkit of elementary rules for computing Laplace transforms.

The *first shifting theorem*, (also called the *first translation theorem*) enables us to calculate Laplace transforms of products of functions of the form $e^{kt} f(t)$, in terms of Laplace transform of $f(t)$. It will also be used in calculating inverse Laplace transforms. It can be stated as

$$L\{e^{kt} f(t)\} = F(s-k) \equiv L\{f(t)\}]_{s \to s-k}$$

Proof

$$\begin{aligned} L\{e^{kt} f(t)\} &= \int_0^\infty e^{-st} e^{kt} f(t)dt \\ &= \int_0^\infty e^{-(s-k)t} f(t)\, dt \\ &= \int_0^\infty e^{-s't} f(t)\, dt \quad \text{where } s' = s-k \\ &= F(s') = F(s-k) \equiv L\{f(t)\}]_{s \to s-k} \end{aligned}$$

Rule of scales

It enables us to calculate Laplace transform of a function of the form $f(at)$ where $a > 0$ is a constant. It states

$$L\{f(at)\} = \frac{1}{a} L\{f(t)\}, \quad a > 0$$

Proof

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty e^{-st} f(at)dt, \quad a0 \\ &= \int_0^\infty e^{-(s/a)t} f(t')\, dt'/a, \quad at = t' \end{aligned}$$

$$= \frac{1}{a}\int_0^\infty e^{-s't'}\, f(t')\,dt', \quad \text{where } s' = s/a$$
$$= \frac{1}{a}\int_0^\infty e^{-s't'}\, f(t')\,dt'$$
$$= \frac{1}{a}L\{f(t)\}, \quad a > 0$$

6.4 Laplace Transforms of Derivatives and Integrals

6.4.1 Laplace transforms of derivatives of a function

The following theorem provides states the necessary conditions ***??for the existence of Laplace transforms of derivatives of a function. **Theorem**

(i) If $f(t)$ is continuous and $f'(t)$ is piecewise continuous on the interval $[0, \infty)$, and both are of exponential order, *i.e.* both of order $e^{\alpha x}$, then

$$\begin{aligned} L\{f'(t)\} &= s\,L\{f(t)\} - f(0) \\ &= sF(s) - f(0) \end{aligned}$$

(ii) If $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on the interval $[0, \infty)$, and all are of exponential order, *i.e.* both of order $e^{\alpha x}$, then

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

(iii) If $f(t)$, $f'(t)$, $f''(t), \cdots f^{n-1}(t)$ are continuous and $f^n(t)$ is piecewise continuous on the interval $[0, \infty)$, and all are of exponential order, *i.e.* both of order $e^{\alpha x}$, then

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Proof

Using definition and integrating by parts

$$L\{f'(t)\} = \int_0^\infty e^{-st}\, f'(t)\,dt$$

$$= e^{-st} f(t)]_0^\infty - (-s)\int_0^\infty e^{-st} f(t)\,dt$$
$$= -f(0) + sL\{f(t)\}$$
$$= sF(s) - f(0)$$

where $F(s)$ is the Laplace transform of $f(t)$.

(ii) Again using definition and the result (i) above, we have

$$L\{f''(t)\} = L\{g'(t)\} \text{ where} g(x) = f'(x)$$
$$= sG(s) - g(0)$$

where we have used result (i) above and $G(s) = L\{g(t)\}$. Now

$$G(s) = L\{g(t)\} = L\{f'(t)\} = s\,F(s) - f(0)$$

Therefore on substitution, we obtain

$$L\{f''(t)\} = s[sF(s) - f(0)] - f'(0)$$
$$= s^2F(s) - sf(0)] - f'(0)$$

(iii) By repeated application of the results (i) or (ii) we can derive this result.

6.4.2 Laplace transform of the integral of a function

Let $g(t) = \int_0^t f(\tau)\,d\tau$, then

$$L\{g(t)\} = \frac{1}{s} L\{f(t)\}$$

To prove this theorem, we note that by fundamental theorem of calculus, *viz.* $g'(t) = f(t)$. Therefore

$$L\{g'(t)\} = sL\{g(t)\} - g(0).$$

But

$$g(0) = \int_0^0 f(\tau)\,d\tau = 0 \text{ and } g'(t) = f(t)$$

Hence

$$L\{g'(t)\} = sL\{g(t)\} \implies L\{g(t)\} = \frac{1}{s} L\{f(t)\}$$

or finally

$$L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s} L\{f(t)\} = \frac{F(s)}{s}.$$