Unit Normal vector to the surface $u_1 = c_1$ at $P = \hat{E}_1$

Unit Normal vector to the surface $u_2 = c_2$ at $P = \hat{E}_2$

Unit Normal vector to the surface $u_3 = c_3$ at $P = \hat{E}_3$

$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|} \qquad \hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|} \qquad \hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$

→ At each point P of a curvilinear coordinate system there exist two sets of unit vectors

(1) $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the coordinate curves.

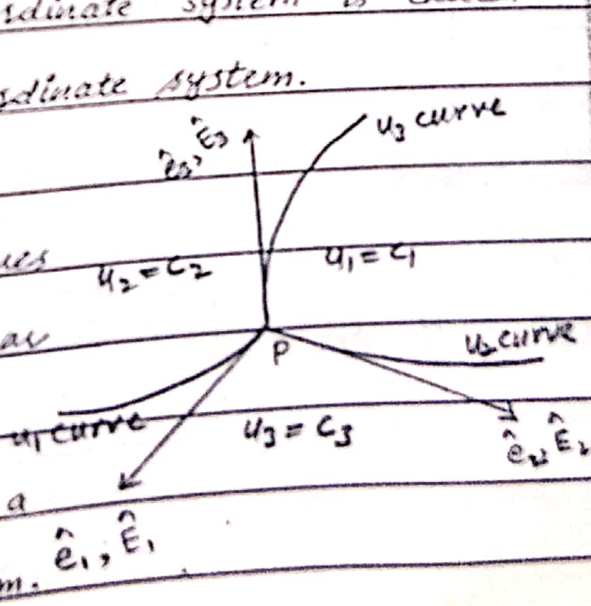
(2) $\hat{E}_1, \hat{E}_2, \hat{E}_3$ Normal to the coordinate surfaces.

→ These two sets of unit vectors generally varry in direction from point to point because coordinate curves are curved.

→ These two sets of unit vectors become identical iff the curvilinear coordinate system is orthogonal.

ORTHOGONAL CURVILINEAR COORDINATE SYSTEM :-

If the coordinate curves intersect at right angles then the curvilinear coordinate system is called orthogonal curvilinear coordinate system.



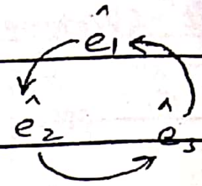
→ u_1, u_2, u_3 coordinate curves of an orthogonal curvilinear system are similar to x, y, z coordinates axes of a rectangular cartesian system.

→ For orthogonal curvilinear coordinate system $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are the same.

→ $\hat{e}_1 \cdot \hat{e}_1 = 1$, $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

$\hat{e}_2 \cdot \hat{e}_2 = 1$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$

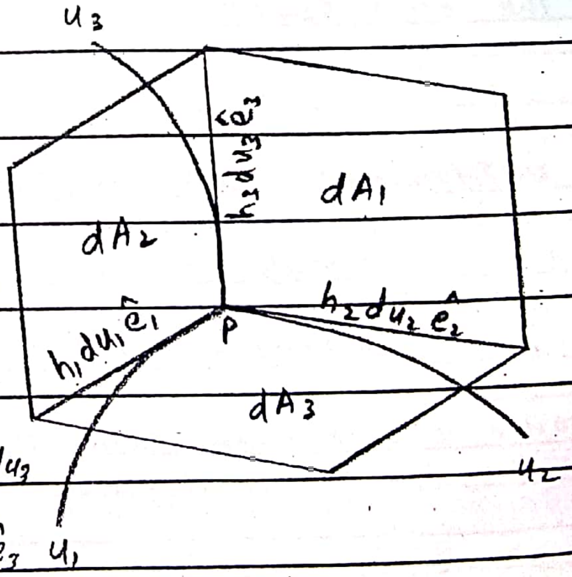
$\hat{e}_3 \cdot \hat{e}_3 = 1$, $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$



→ In an orthogonal curvilinear coordinate system vector $\vec{A}$ can be expressed in terms of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

EXPRESSIONS FOR ARC LENGTH, AREA, AND VOLUME ELEMENTS IN ORTHOGONAL CURVILINEAR COORDINATES:



i) ARC LENGTH ELEMENT:

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$d\vec{r} = (h_1\hat{e}_1)du_1 + (h_2\hat{e}_2)du_2 + (h_3\hat{e}_3)du_3$$

$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$(ds)^2 = d\vec{r} \cdot d\vec{r}$$

$$(ds)^2 = h_1^2 du_1^2 (\hat{e}_1 \cdot \hat{e}_1) + h_2^2 du_2^2 (\hat{e}_2 \cdot \hat{e}_2) + h_3^2 du_3^2 (\hat{e}_3 \cdot \hat{e}_3)$$

$(ds)^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$ $\quad \because \hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$

$ds = \sqrt{h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2}$

is the element of arc length ds.

→ Along u_1-curve:

u_2, u_3 are constants $\Rightarrow d\vec{r} = h_1 du_1 \hat{e}_1$

$\Rightarrow ds_1 = h_1 du_1$

→ Along u_2-curve:

u_1, u_3 are constant $\Rightarrow d\vec{r} = h_2 du_2 \hat{e}_2$

$\Rightarrow ds_2 = h_2 du_2$

→ Along u_3-curve:

u_1, u_2 are constants $\Rightarrow d\vec{r} = h_3 du_3 \hat{e}_3$

$\Rightarrow ds_3 = h_3 du_3$

ii) AREA ELEMENT:

From Figure

$dA_1 = |(h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)|$

$dA_1 = h_2 h_3 du_2 du_3 |(\hat{e}_2 \times \hat{e}_3)|$

$dA_1 = h_2 h_3 du_2 du_3 |\hat{e}_1|$

$\because \hat{e}_1 =$ unit vector

$dA_1 = h_2 h_3 du_2 du_3 (1)$

$dA_1 = h_2 h_3 du_2 du_3$

similarly $dA_2 = h_1 h_3 du_1 du_3$

$dA_3 = h_1 h_2 du_1 du_2$

Hence

$$\boxed{\begin{aligned} dA_1 &= h_2 h_3\, du_2\, du_3 \\ dA_2 &= h_1 h_3\, du_1\, du_3 \\ dA_3 &= h_1 h_2\, du_1\, du_2 \end{aligned}}$$

iii) VOLUME ELEMENT :

The absolute value of the scalar triple product gives the volume of the parallelopiped.

$$dV = |(h_1 du_1 \hat{e}_1) \cdot (h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)|$$

$$dV = (h_1 du_1)(h_2 du_2)(h_3 du_3)\ |\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3|.$$

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3\ |\hat{e}_1 \cdot \hat{e}_1| \qquad \because \hat{e}_2 \times \hat{e}_3 = \hat{e}_1$$

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3\ |1| \qquad \because \hat{e}_1 \cdot \hat{e}_1$$

$$\boxed{dV = h_1 h_2 h_3\, du_1\, du_2\, du_3}$$

volume