LINEAR COMBINATION

General Point Of View:

In linear algebra we say linear combinations in terms of vectors. But, in general we can say that linear combinations of anything can be done

(Scalar) (something1) + (scalar) (something2) + (scalar) (something3)

This something can be anything like x or y or z etc (3x + 2y is a linear combination of x and y with scalars 3 and 2 respectively) or matrices or something more complicated like polynomial. In general, a linear combination is a particular way of combining things (variables, vectors, etc) using scalar multiplication and addition.

Linear Combination is obtained by the multiplying matrices by scalars, and adding them together. So simply in this method we use multiplication and addition. One thing important is that the dimension of the matrices should be same for linear combination if we applying linear combination on more than one matrix. The concept of linear combination is very important in Linear Algebra.

Definition:

If A be a square matrix of same size and x be a vector (n×1) column vector as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then their Liner Combination will be:

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots + x_3 \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

If we defined in the form of Vector:

Suppose we have a field (for example real number) we denoted it by C and vector space V. If vectors $v_1, v_2, \ldots v_n$ *in* R^n *and scalars* $c_1, c_2 \ldots c_n$ *then Linear Combination of those vectors with those scalars is given by:*

$B = c_1v_1 + c_2v_2 \ldots c_nv_n$.

Applications:

The concept of linear combination is central to linear algebra.

1) *Most of this article deals with linear combinations in the context of a vector space.*
2) *An important application of linear combinations is to wave functions in quantum mechanics.*
3) *It is also used when we have to find the combination of atoms and molecules.*

EXAMPLES:

1) ***Here we take example of linear combination for one matrix.***

Let suppose we have a matrix A and vector x, given as:

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 4 \\ 3 & 0 \end{bmatrix} \text{ and } \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Then their Linear Combination will be

$$Ax = 2\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 12 \\ 0 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 18 \\ 14 \\ 6 \end{bmatrix}$$

This is the Linear Combination of matrix A and vector x.

2) ***Now we take the example of Linear Combination for Vector and scalars.***

Consider we have two vector $v_1 = (3, 3, 1)$ *and* $v_2 = (1, 3, 2)$ *and scalars* $a_1 = 2$, $a_2 = 3$ *then their Linear Combination will be*

$b = a_1 v_1 + a_2 v_2$

$b = 2 (3, 3, 1) + 3 (1, 3, 2)$

b = (6, 6, 2) + (3, 9, 6)

b = (9, 15, 8)

This is Linear Combination of the vectors and scalars.

Real Life Example

Three people denoted by P1, P2, P3 intend to buy some rolls, buns, cakes and bread. Each of them needs these commodities in differing amounts and can buy them in two shops S1, S2. Which shop is the best for every person P1, P2, P3 to pay as little as possible? The individual prices and desired quantities of the commodities are given in the following tables.

Demand quantity of Foodstuff

	Roll	Bun	Cake	Bread
P_1	6	5	3	1
P_2	3	6	2	2
P_3	3	4	3	1

Price in shop S_1 and S_2

	S_1	S_2
Roll	1.50	1.00
Bun	2.00	2.50
Cake	5.00	4.50
Bread	16.00	17.00

Now we make matrices of above tables

$$P = \begin{bmatrix} 6 & 5 & 3 & 1 \\ 2 & 6 & 2 & 2 \\ 3 & 4 & 3 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1.50 & 1.00 \\ 2.00 & 2.50 \\ 5.00 & 4.50 \\ 16.00 & 17.00 \end{bmatrix}$$

Now we find the Linear Combination of the above. Multiply the each column of Q with P as a Linear Combination.

$$1.50\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} + 2.00\begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} + 5.00\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + 16\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 4.5 \end{bmatrix} + \begin{bmatrix} 10 \\ 12 \\ 8 \end{bmatrix} + \begin{bmatrix} 15 \\ 10 \\ 15 \end{bmatrix} + \begin{bmatrix} 16 \\ 32 \\ 16 \end{bmatrix} = \begin{bmatrix} 50 \\ 57 \\ 43.5 \end{bmatrix}$$

$$1.00\begin{bmatrix}6\\2\\3\end{bmatrix}+2.50\begin{bmatrix}5\\6\\4\end{bmatrix}+4.50\begin{bmatrix}3\\2\\3\end{bmatrix}+17\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}6\\2\\3\end{bmatrix}+\begin{bmatrix}12.5\\15\\10\end{bmatrix}+\begin{bmatrix}13.5\\9\\13.5\end{bmatrix}+\begin{bmatrix}17\\34\\17\end{bmatrix}=\begin{bmatrix}49\\60\\43.5\end{bmatrix}$$

$$PQ=\begin{bmatrix}50 & 49\\57 & 60\\43.5 & 43.5\end{bmatrix}$$

Result:

P_1 spend 50 and 49 in S_1 and S_2 respectively.

P_2 spend 57 and 60 in S_1 and S_2 respectively.

P_3 spend 43.5 in both S_1 and S_2 respectively.

Question

Now we have a given question to solve in which we have to take a matrix order of (6 × 6) and a vector of order (6 × 1) and find the possible combination of matrix.

Let we have matrix A = $\begin{bmatrix}1 & 2 & 7 & 3 & 4 & 5\\5 & 2 & 1 & 9 & 5 & 5\\1 & 3 & 7 & 5 & 2 & 9\\2 & 2 & 1 & 0 & 8 & 8\\3 & 4 & 2 & 0 & 1 & 5\\8 & 1 & 7 & 2 & 3 & 3\end{bmatrix}$ *and vector x =* $\begin{bmatrix}5\\2\\3\\4\\1\\1\end{bmatrix}$,

Now the possible combination of matrix will be given by ***linear Combination*** *we now know that by linear combination we find the combination of matrix and vector* ***(We already explain this in above discussion that why we use this method to find the combination of given matrix).***

In this we multiply the each element of vector x to the column of matrix A.

Let Ax show the linear combination of A and x, then according to definition of the linear combination,

$$Ax = 5\begin{bmatrix}1\\5\\1\\2\\3\\8\end{bmatrix} + 2\begin{bmatrix}2\\2\\3\\2\\4\\1\end{bmatrix} + 3\begin{bmatrix}7\\1\\7\\1\\2\\7\end{bmatrix} + 4\begin{bmatrix}3\\9\\5\\0\\0\\2\end{bmatrix} + 1\begin{bmatrix}4\\5\\2\\8\\1\\3\end{bmatrix} + 1\begin{bmatrix}5\\5\\9\\8\\5\\3\end{bmatrix}$$

By multiplying the vector elements to the matrix's columns

$$Ax = \begin{bmatrix}5\\25\\5\\10\\15\\40\end{bmatrix} + \begin{bmatrix}4\\4\\6\\4\\8\\2\end{bmatrix} + \begin{bmatrix}21\\3\\21\\3\\6\\21\end{bmatrix} + \begin{bmatrix}12\\36\\20\\0\\0\\8\end{bmatrix} + \begin{bmatrix}4\\5\\2\\8\\1\\3\end{bmatrix} + \begin{bmatrix}5\\5\\9\\8\\5\\3\end{bmatrix}$$

By adding them together we obtain

$$Ax = \begin{bmatrix}51\\78\\63\\33\\35\\77\end{bmatrix}$$

This is the Linear Combination of the matrix A and vector x. And by above method we obtain the combination for the given matrix A.

Question

▶ EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

▶ EXAMPLE 9 Columns of a Product *AB* as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

It follows from Formula (6) and Theorem 1.3.1 that the jth column vector of AB can be expressed as a linear combination of the column vectors of A in which the coefficients in the linear combination are the entries from the jth column of B. The computations are as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$ ◀

pansion

Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

Column-Row Expansion Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

has size $m \times n$ so the sum itself is an $m \times n$ matrix. We leave it as an exercise for you to verify that the entry in row i and column j of the sum is given by the expression on the right side of Formula (5), from which it follows that

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r \tag{11}$$

We call (11) the ***column-row expansion*** of AB.

EXAMPLE 10 Column-Row Expansion

Find the column-row expansion of the product

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} \tag{12}$$

Solution The column vectors of A and the row vectors of B are, respectively,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \quad \mathbf{r}_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$