Historical Note

In 1810, Gauss sets out to determine details about the orbit of the **Pallas,** the second largest asteroid of the solar system. He obtains a system of linear equations in six unknowns, where not all equations can be satisfied simultaneously. Hence he needs to determine values for the unknowns will minimize the total square. Instead of merely solving the problem at hand, Gauss disagrees and introduces a method for dealing with such systems of linear equations in general. This is where his characteristics notation appears for first time. Gauss now uses a procedure that is essentially equivalent to what today is known as Gaussian elimination.

Some authors use the term Gaussian elimination to refer only to the procedure until the matrix is in echelon form, and use the term Gauss Jordan elimination to refer to the procedure which ends in reduced echelon form. The name used because it is a variation in Gaussian elimination as described by William Jordan in 1888. However, the method also appears in an article by Clasen published in same year. Jordan and Clasen probably discovered Gauss Jordan elimination independently.

Now I explain some terms used in this method.

Concept of echelon and reduced echelon form

Echelon Form

In echelon form means that the matrix meets the following three requirements.

- The first number in the row is 1.
- Every leading entry 1 is to the right of the one above it.
- Any non-zero rows are always above with all zeros.

Here is an example of echelon form,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced Echelon form

A matrix is in reduced echelon form it meets the following requirements.

- The first non-zero number in the first row (leading entry) is 1.
- The second row also starts with the number 1, which is further to right than leading entry in the first row.
- The leading entry in each row must be the only non-zero number in its column.
- Any row with all entries zero is at the bottom of the matrix.

Here is an example of reduced echelon form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Difference in echelon and reduced echelon form mathematically

Generally there in no difference in echelon and reduced echelon form. In echelon form we find unknowns by using little bit algebra but in reduced echelon form we find values directly by back substitution. Let us explain it with following example.

$$x + y - 2z = -2$$

$$y + 3z = 7$$

$$x - z = -1$$

Echelon Form

Matrix form of above equation is,

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

It is already in echelon form by back substitution,

$$x + y - 2z = -2, y + 3z = 7, z = 2$$

After simplifying we get unknowns,

$$x = 1, y = 1, z = 2$$

Reduced Echelon Form

By applying operations we get reduced echelon form as,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ Reduced echelon form is usually identity.

Using back substitution we get unknowns,

$$x = 1, y = 1, z = 2$$

We can see here that both methods give the same result.

Where Gauss Jordan Method fails

The Gaussian elimination method fails if one of pivot elements becomes zero. In such a situation, we rewrite the equation in a different order to avoid zero pivots. Changing the order of equations is called pivoting.

What is pivot element?

The pivot element is the element of a matrix or an array, which is selected first by an algorithm (Gaussian elimination) to do certain calculation. Let us explain it with following example,

Let us any take an augmented matrix in row scenario.

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 2 & 0 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

According to algorithm, first entry should be non-zero. In this situation, we easily get rid of this problem by pivoting (exchanging row). But when all first elements are zero pivoting method also fails then we use Gauss Jordan elimination method to solve system of equations. Generally Gauss Jordan method puts matrix in identity matrix. When matrix is not identity than system has infinite many solutions. For example,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

When we look at first case where (0=0), this means it has infinite many solutions. But when we meet with this situation where (0=2), this is impossible, hence method fails.

This method does not work also when matrix is singular ($|A| = 0$).

Note: Gaussian elimination helps to put matrix in echelon form and Gauss Jordan elimination method puts matrix in reduced echelon. For small system it is usually more convenient to use Gauss Jordan elimination method and explicitly solve for each variable in the matrix system.

Applications of Gauss Jordan Elimination Method

Historically, the first application is for solving system of linear equations. Here are some important applications of Gauss Jordan elimination.

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- Computation of inverse of a matrix by "sorting" the elementary row operations over the identity matrix.

- Computing determinants.

- Computing ranks and bases.

Calculating Inverse Using Gauss Jordan Elimination Method

Why we need inverse of matrix

Before finding inverse, we need to know why do we need an inverse. Because with matrices we don't divide, there is no concept of dividing by a matrix. But we can multiply by an inverse, which achieves the same thing. Inverse matrix used in cryptography applications where we encrypt messages by multiplying them with matrix and we need to know the inverse to decode.

Method for Finding Inverse

The Gauss Jordan method allows us to calculate the inverse of a matrix by performing elementary row operations between its rows using this procedure.

$$(A \mid I) \rightarrow (I \mid A^{-1})$$

Here A is given matrix whose inverse is desired and I is identity matrix called augmented matrix. Using algorithm, we apply elementary row operations on matrix A to transform it into identity matrix (reduced echelon form) and same operations must be applied on identity (augmented) matrix. The matrix A is invertible if and only if the left block can be reduced to the identity matrix, in this case the right block of final matrix is A^{-1}.

Checking the Inverse Calculation

We are going to check that we have calculated the inverse correctly. To know if it is ok, we have to multiply the original matrix by the inverse matrix and result is identity matrix.

$$A \cdot A^{-1} = I$$

Determinant of Inverse Using Gauss Jordan Elimination Method

Suppose A is an invertible matrix. Then there exists some inverse A^{-1} such that $A \cdot A^{-1} = I$. Where I is the identity matrix? It follows that $\det(A \cdot A^{-1}) = \det(I)$. Making use of this fact that the determinant of the product of two matrices is just the product of the determinants, and determinant of identity matrix is 1, we get

$$\det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Determinants are just numbers and by necessity, neither determinant is zero.

Why we use this method?

Finding the inverse of a 2×2 matrix is a simple task, but for finding the inverse of large matrix $3 \times 3, 4 \times 4$ etc. is tough task, So the following methods can be used:

- Elementary row operation (Gauss Jordan Method) **(Efficient)**
- Minors, Cofactors and ad-joint Method **(Inefficient)**

Example: Find Determinant of inverse using Gauss Jordan method.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{bmatrix}$$

Solution

We start with the matrix A, and write it down with an identity matrix | next to it:

$$=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

This is called “Augmented Matrix”

$$=\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \quad (-1)\,R_1$$

$$=\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & -2 & -4 & -3 & -1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_2 \to 3R_1 - R_2, R_3 \to R_1 + R_3$$

$$=\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & -3/2 & -1/2 & 0 \\ 0 & 1 & 1 & -1/2 & 0 & 1/2 \end{array}\right] \quad R_2 \to R_2/2, R_3 \to R_3/2$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & -1 & -2 & -3/2 & -1/2 & 0 \\ 0 & 0 & -1 & -2 & -1/2 & 1/2 \end{array}\right] \quad R_1 \to R_1 - R_2, R_3 \to R_2 + R_3$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & -1 & 0 & 5/2 & 1/2 & -1 \\ 0 & 0 & -1 & -2 & -1/2 & 1/2 \end{array}\right] \quad R_2 \to R_2 - 2R_3$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & -5/2 & -1/2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 & -1/2 \end{array}\right] \quad R_2 \to (-1)R_2, R_3 \to (-1)R_3$$

The right block is the inverse of given matrix.

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & -1 \\ 2 & 1/2 & -1/2 \end{bmatrix}$$

Checking the Inverse

As we discussed earlier that we can check whether we find inverse correctly or not, we multiply inverse by original matrix and result will be identity matrix.

$$A \cdot A^{-1} = I$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ 2 & 1/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}+\frac{3}{2} & \frac{1}{2}-\frac{1}{2} & 1-1 \\ \frac{5}{2}-\frac{3}{2}-1 & -\frac{5}{2}+\frac{1}{2}+3 & -5+1+4 \\ -2+\frac{3}{2}+\frac{1}{2} & 2-\frac{1}{2}-\frac{3}{2} & 4-1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

The result is identity so, our inverse is correct.

Determinant of Inverse

Using above property,

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

We have to calculate determinant of A first,

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{bmatrix}$$

$$|A| = -1\begin{vmatrix} -1 & -2 \\ 3 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} +2\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$=-1(-4+6)-1(12-2)+2(9-1)$$

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$$=-2-10+16$$

$$=4$$

So, by definition determinant of inverse is,

$$\det\left(A^{-1}\right)=\frac{1}{4}$$

Applications of Inverse Matrices

- The use of coding becomes significant in recent years. One way to encrypt or code a message using matrices and their inverse. Consider a fixed invertible matrix A. Convert the message into a matrix B such that AB is possible to perform. The message which we sent is AB. At the other end, we need to know A^{-1} in order to decrypt or decode the message sent.

$$A^{-1}(AB)=B$$

Where B is the original message.

- The following 2 by 2 anti-diagonal matrix with 0 and 1 having determinant -1 being inverse of itself. This matrix plays the role of NOT gate or state inversion in digital electronics and can be extended to any order.

Charge density J_O and J_I can be written as 2 by 1 column matrix.

$$\begin{bmatrix} J_O \\ J_I \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} J_O \\ J_I \end{bmatrix}$$

$$\begin{bmatrix} J_O \\ J_I \end{bmatrix}=\begin{bmatrix} J_I \\ J_O \end{bmatrix}$$

$$J_O=J_I$$

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Charge density is transformed into current density. And current density converted into charge density.

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Elimination method

The elimination method of solving systems of equations is also called the addition method. To solve a systems by elimination we transform the system such that one variable "cancel out". In order to solve for y, take the value for x and substitute it back into either one of the original equations.

In elimination method we either add or subtract the equations to get an equation in one variable.

Example

$$3y + 2x = 6$$
$$5y - 2x = 10$$

We can eliminate the x-variable by addition of two equations.

$$\begin{array}{r} 3y + 2x = 6 \\ + \underline{5y - 2x = 10} \\ 8y = 16 \end{array} \Rightarrow y = 2$$

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$$2x - 3(-4x + 24) = -2$$
$$2x + 12x - 72 = -2$$
$$14x = 70$$
$$x = 5$$

Plug value of x into origional equation for y,

$$y = -4(5) + 24$$
$$= -20 + 24$$
$$y = 4$$

This method also works when we have two unknown variables. It is difficult and time ~~so~~ consuming for more variable. For large system we introduce another method called Gauss elimination method.

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Substitution method

The method of solving "by substitution" works by solving one of the equations (we choose which one) for one of the variables and then plugging this back into other equation, "substituting" for the chosen variable and solving for the other. Then we back-solve for the first variable

Example

$$2x - 3y = -2$$
$$4x + y = 24$$

The idea here is to solve one of the equations for one of the variables, and plug this into other equation.

Solving second equation for y:

$$y = -4x + 24$$

Plug this in first equation, and solve for x:

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The value of y can now be substituted into either of the origional equations to find the value of x.

$$3y + 2x = 6$$
$$3.2 + 2x = 6$$
$$6 + 2x = 6$$
$$2x = 0 \implies x = 0$$

If we don't have equations where we can eliminate a variable by addition or subsbraction, we directly can begain by multiplying one or both the equations with a constant to obtain an equivalent linear system where we can eliminate one of the variables by addition or subsbraction.

This method is efficient when we have only two equations or it may work when three equations present but for three equations it is time taking process.

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Gauss Elimination method.

Gauss elimination method is used when we have large system of equations. Its algorithm is to apply elemantry row operations to change matrix in a row echelon form and apply back substitution to find unknowns. Let we have system of equation,

$$x + 3y - z = 4$$
$$3x - y + 3z = 3$$
$$-x + 4y - z = -2$$

Its matrix form "called augmented matrix" is,

$$\left[\begin{array}{ccc:c} 1 & 3 & -1 & 4 \\ 3 & -1 & 3 & 3 \\ -1 & 4 & -1 & -2 \end{array}\right]$$

called augmented matrix

Applying elemantry row operations we change augmented matrix into echelon form and using back substitution we can find unknown variables.

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$$= \begin{bmatrix} -12 & 1.5 & 7 \\ 11 & -5 & 7 \end{bmatrix}$$

Gauss Elimination & Back Substitution

1. $4x - 6y = -11$

$-3x + 8y = 10$

Solution:

In matrix form.

$$\left[\begin{array}{cc:c} 4 & -6 & -11 \\ -3 & 8 & 10 \end{array}\right]$$

Apply $R_2 \to R_2 + \frac{3}{4}R_1$

$$= \left[\begin{array}{cc:c} 4 & -6 & -11 \\ 0 & 7/2 & 7/4 \end{array}\right]$$

~~by~~ Back substitution

$4x - 6y = -11$ $\qquad$ $\frac{7}{2}y = \frac{7}{4}$

$4x = -11 + 3$ $\qquad$ $\boxed{y = 1/2}$

$\boxed{x = -2}$

2. $$\begin{bmatrix} 3 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6.0 \end{bmatrix}$$

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$x+y-z=9$

$8y+6z=-6$

$-2x+4y-6z=40$

Solution

Its matrix form

$$=\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ -2 & 4 & -6 & : & 40 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$=\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 6 & -8 & : & 58 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 0 & -12.5 & : & 62.5 \end{bmatrix} \quad R_3 \to R_3 - \frac{3}{4}R_2'$$

Back substitution

$-12.5z = 62.5$, $8y+6z=-6$

$z=-5$

$8y = 24$

$y=3$

$x+y-z=9$

$x=1$

$\Rightarrow x=1, \ y=3, \ z=-5$

$$\begin{bmatrix} 2 & 4 & 1 & : & 0 \\ -1 & 1 & -2 & : & 0 \\ 4 & 0 & 6 & : & 0 \end{bmatrix}$$

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$x + y - z = 9$

$8y + 6z = -6$

$-2x + 4y - 6z = 40$

Solution

Its matrix form

$$= \begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ -2 & 4 & -6 & : & 40 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 6 & -8 & : & 58 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 & : & 9 \\ 0 & 8 & 6 & : & -6 \\ 0 & 0 & -12.5 & : & 62.5 \end{bmatrix} \quad R_3 \to R_3 - \frac{3}{4} R_2$$

Back substitution

$-12.5z = 62.5$, $8y + 6z = -6$

$z = -5$

$8y = 24$

$y = 3$

$x + y - z = 9$

$x = 1$

$\Rightarrow x = 1$, $y = 3$, $z = -5$

$$\begin{bmatrix} 2 & 4 & 1 & : & 0 \\ -1 & 1 & -2 & : & 0 \\ 4 & 0 & 6 & : & 0 \end{bmatrix}$$

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Solution:

$$= \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 3 & -1.5 & 0 \\ 4 & 0 & 6 & 0 \end{array}\right] \quad R_2 \to R_2 - \frac{1}{2} R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 3 & -1.5 & 0 \\ 0 & -8 & 4 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 3 & -1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - \frac{2}{3} R_2$$

Back substitution

$2x + 4y + z = 0$

$2y - z = 0$

Put $y = t$

$2x = -4y - z$

$2t = z$

$= \dfrac{-4t - 2t}{2}$

$z = 2t$

$x = -3t$

Hence, $x = -3t$, $y = t$, $z = 2t$

$-2y - 2z = -8$

$3x + 4y - 5z = 13$

Solution:

$$\left[\begin{array}{ccc|c} 0 & -2 & -2 & 8 \\ 3 & 4 & -5 & 13 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & -5 & 13 \\ 0 & -2 & -2 & 8 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

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$$= \begin{bmatrix} 3 & 4 & -5 & : 13 \\ 0 & -1 & 1 & : 4 \end{bmatrix} \quad R_2 \to -0.5 R_2$$

$3x + 4y - 5z = 13$ $\quad y + z = 4$

$P_{u}1 \quad z = 1$

$y = -1 + 4$

$$x = \frac{13 - 4y + 5z}{3}$$

$$= \frac{13 - 4(4 - 1) + 5(1)}{3} = \frac{-d3 + 71}{3}$$

$x = 31 - 1$

$x = 31 - 1, \; y = 4 - 1 \;, z = 1$

Gauss Elimination with no solution

$$\begin{bmatrix} 3 & 2 & 1 & : 3 \\ 2 & 1 & 1 & : 0 \\ 6 & 2 & 4 & : 6 \end{bmatrix}$$

Solution:

$$= \begin{bmatrix} 3 & 2 & 1 & : 3 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & : -2 \\ 6 & 26 & 4 & : 6 \end{bmatrix} \quad R_2 \to R_2 - \frac{2}{3} R_1$$

$$= \begin{bmatrix} 3 & 2 & 1 & : 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$= \begin{bmatrix} 3 & 2 & 1 & : 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & : -2 \\ 0 & 0 & 0 & : 12 \end{bmatrix} \quad R_3 \to R_3 - 6R_2$$

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Determinant of the inverse By Gauss-Jordan Method. why we use this method?

Given matrices is

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{pmatrix}$$

∴ Explanation :-

Gauss - Jordan Elimination

This method is an Algorithm that can be used to solve the systems of linear equations and to find the inverse of any invertible matrix. It relies upon three elementary Rows operation one can use on a matrix:

1. swap the Position of two of the Rows.
2. Multiply one of the rows by a non-zero scalar,

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(3) Add or subtract the scalar multiple of one row to another Row.

For example :-

The first elementary Row operation, swap the positions of the 1st and 3rd row

$$\begin{pmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 7 & 5 & 0 \\ 2 & -2 & 3 \\ 4 & 0 & -1 \end{pmatrix}$$

For an example of second Elementary Row operation, multiply the second row by 3.

$$\begin{pmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 0 & -1 \\ 6 & -6 & 9 \\ 7 & 5 & 0 \end{pmatrix}$$

For an example of 3rd Elementary Row operation, Add twice the 1st row to the 2nd Row.

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$$\begin{pmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 0 & -1 \\ 10 & -2 & 1 \\ 7 & 5 & 0 \end{pmatrix}$$

- **Reduced - Row echlon form:**

Purpose of Gauss-Jordan Elimination

The purpose of this elimination is to use the three elementary row operations. to convert a matrix into reduced - row echlon form. A matrix is in reduced. Row echelon form, also known as row canonical form, if the following conditions are satisfied.

- 1 All Rows with only zero entries are at the bottom of the Matrix
- 2. The first non-zero entry is a in a row, called leading entry or the pivot, of each non-zero row is to the right of the leading entry of the row above it.

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- 3 – The leading entry also known as the pivot, in any non-zero row is 1
- 4 – All other entries in the column containing a leading 1 are zeros (0)

For example :-

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 7 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 7 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Steps for Gauss - Jordan Elimination

To perfor Gauss - Jordan elimination following steps are followed

1. swap the rows so that all rows with all zero entries are on the bottom.

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2 — swap the rows so that the rows with the largest, leftmost non-zero entry is on top.

3 — Multiply the top Row by a scalar so that top Rows leading entry becomes 1.

4 — Add / subtract multiplies of the top row to the other rows so that all other entries in the column containing the top row's leading entry are all zero.

5 — we use only Row-operations instead of coloum operations

6 — The left-side entries are at last convert into upper and lower triangles. Echlon and Reduce Echelon form steps are envolved simultaneously.

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Basic Concept :-

The basic concept of ~~F~~ Gauss-Jordan Method is that we suppose given a Matrix A. And we have to find the Inverse of this matrix Than

- First I made its Augmented form.

$$[A : I]$$

- we write 3×3 order Identity Matric with given matrix in such a form.
- Now we multiply them with inverse of Matrix

$$[A \times A^{-1} : I \times A^{-1}]$$

$$= [I : A^{-1}]$$

So, gives matrix replace the Identity Matrix and Identity Matrix Replace the Inverse of Matrix.

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Other methods to find A^{-1}

1 Gauss's Elimination method.

In this method Three Equations are given and we made their Augmented form and than find their solution for the value of x, y and Z.

- **By Adjoint Method :-**

We find inverse by Adjoint method.

$$A^{-1} = \frac{Adj(A)}{|A|}$$

This method is used in earlier classes.

These both Methods are difficult and not exact methods and many other more other methods are used. But this method is very exact and simple method.

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→ Inverse of the given Matrix by Gauss Jordan Method.

We have given a Matrix A and our goal is to find the Inverse of it.

Given Matrix :-

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{bmatrix}$$

→ Solution :-

- In Gauss-Jordan Method: we know that we have to apply operations on the Rows not on the columns.
- We know that

$$= [A : I]$$

$$= [A.A^{-1} : I.A^{-1}] \quad \times \text{ by } A^{-1}$$

$$= [I : A^{-1}]$$

- First write its Augmented Matrix. than relate ~~and~~ the Identity and original Matric. we can Also Apply division on Any Row.

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• Step : 1 :–

Relate given Matrices with the Identity Matrix of Order 3 × 3.

$$[A \; : \; I]$$

$$\sim \left[\begin{array}{ccc:ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

• Step — 2

In Order to apply Gauss–Jordan elimination we need first term (+ve) and equal to One, so we apply division operation on Row one and Row 2

$$R_1/_{-1} \longrightarrow R_1 \quad \& \quad R_3/_{-1} \longrightarrow R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ +3 & -1 & -2 & 0 & +1 & 0 \\ 1 & -3 & -4 & 0 & 0 & -1 \end{array}\right]$$

⇒ Now, we have to apply only Row operations.

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Step — 3

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & +2 & +4 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 & 0 & 1 \end{array}\right]$$

In this step, with the help of 1st one entry. we make zeros under this entry.

Step — 4

Now with the help of central element (2) we make both up and down zero (0).

operation:

$$R_1 \rightarrow 2R_1 + R_2$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 1 & -1 \end{array}\right]$$

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• Step — 5 Apply operation

$R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & -5 & -1 & 2 \\ 0 & 0 & 2 & 4 & 1 & -1 \end{array}\right]$$

• Step — 6

In order to make first Matrices Identity we have to devide three Rows with 2

$R_1 \longrightarrow R_1/2$

$R_2 \longrightarrow R_2/2$

$R_3 \longrightarrow R_3/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & -5/2 & -1/2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 & -1/2 \end{array}\right]$$

• Step — 7

compare the step-6 with

$[I : A^{-1}]$

so, we find its Inverse matrix of given Matrices

Which is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ 2 & 1/2 & -1/2 \end{bmatrix}$$

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Now check weather the Answer is Right or wrong.
We Apply this Relation

$$A \cdot A^{-1} = I$$

Multiply the original Matrices with the Inverse Matrix. If Answer is identity than Inverse is Correct.

$$L.H.S = A \cdot A^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & -2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ 2 & 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} - 5/2 + 4 & -\frac{1}{2} - \frac{1}{2} + 1 & 0 + 1 - 1 \\ 3/2 + 5/2 - 4 & 3/2 + \frac{1}{2} - 1 & 0 - 1 + 1 \\ -\frac{1}{2} - \frac{15}{2} + 8 & -\frac{1}{2} - 3/2 + 2 & 0 + 3 - 2 \end{bmatrix}$$

After calculations

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R.H.S$$

Determinant of Inverse Matrix:

We know that the

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ 2 & 1/2 & -1/2 \end{bmatrix}$$

In order to find Determinant

$$|A^{-1}| = \begin{vmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ 2 & 1/2 & -1/2 \end{vmatrix}$$

Expand this with 1st Row

$$= \frac{1}{2}\begin{vmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} -5/2 & 1 \\ 2 & -1/2 \end{vmatrix} + 0\begin{vmatrix} -5/2 & -1/2 \\ 2 & 1/2 \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{5}{4} - 2\right) + 0$$

$$= \frac{1}{2}\left(-\frac{1}{4}\right) - \frac{1}{2}\left(\frac{-3}{4}\right)$$

$$= -\frac{1}{8} + 3/8$$

$$|A^{-1}| = \boxed{1/4} \text{ Ans}$$

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Why we use this Method?

- Gauss-Jordan Method is very useful technique in solving the system of linear Equation.

- This method is quite useful when Equations are Three or more than Three. other methods like Gauss's elimination method are not easy to use for more than Three equations. And Adjoint method is also quite Difficult to use.

- Gauss-Jordan put the Matrix Eachlon and Reduced echlon form At a time simultaneously. Therefore we can use it to find the exact values.

- In Gauss elimination three equations are obtained and we have to them solve it but in this exact value is obtained:

$$[A : I] \quad \times \text{by } A^{-1}$$

$$[I : A^{-1}]$$

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❖ Important Results :-

⇒ There are many other methods which are used to Determine the Inverse of a Matrix but this method is very useful and easy to Apply and Very Direct Method.

⇒ In this method No more calculations are needed after applying the Row operations on the Matrices.

⇒ The final and important Result which we obtained after such a calculation is the Inverse of the given Matrix is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -5/2 & -1/2 & 1 \\ -2 & 1/2 & -1/2 \end{bmatrix}$$

The End.........

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