4.5 Thevenin's Theorem

It often occurs in practice that a particular element in a circuit is variable (usually called the *load*) while other elements are fixed. As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load. Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

According to Thevenin's theorem, the linear circuit in Fig. 4.23(a) can be replaced by that in Fig. 4.23(b). (The load in Fig. 4.23 may be a single resistor or another circuit.) The circuit to the left of the terminals *a*-*b* in Fig. 4.23(b) is known as the *Thevenin equivalent circuit*; it was developed in 1883 by M. Leon Thevenin (1857–1926), a French telegraph engineer.

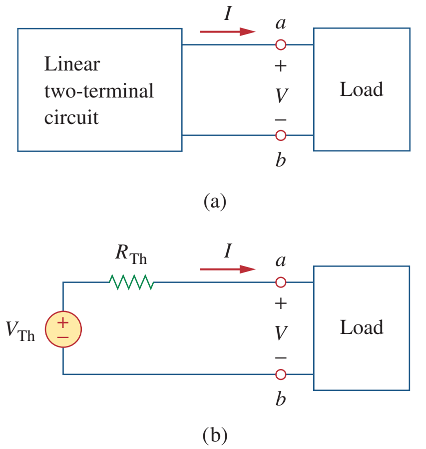


Figure 4.23
Replacing a linear two-terminal circuit by its Thevenin equivalent: (a) original circuit, (b) the Thevenin equivalent circuit.

> **Thevenin's theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.

The proof of the theorem will be given later, in Section 4.7. Our major concern right now is how to find the Thevenin equivalent voltage V_{Th} and resistance R_{Th}. To do so, suppose the two circuits in Fig. 4.23 are equivalent. Two circuits are said to be *equivalent* if they have the same voltage-current relation at their terminals. Let us find out what will make the two circuits in Fig. 4.23 equivalent. If the terminals *a*-*b* are made open-circuited (by removing the load), no current flows, so that the open-circuit voltage across the terminals *a*-*b* in Fig. 4.23(a) must be equal to the voltage source V_{Th} in Fig. 4.23(b), since the two circuits are equivalent. Thus V_{Th} is the open-circuit voltage across the terminals as shown in Fig. 4.24(a); that is,

$$V_{Th} = v_{oc} \tag{4.6}$$

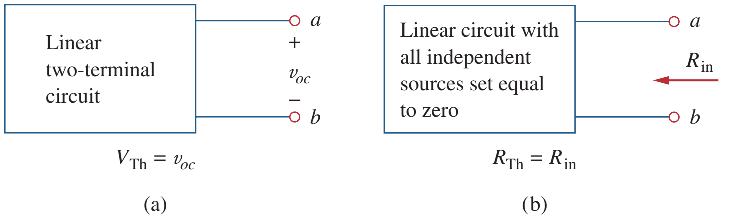


Figure 4.24
Finding V_{Th} and R_{Th}.

Again, with the load disconnected and terminals *a*-*b* open-circuited, we turn off all independent sources. The input resistance (or equivalent resistance) of the dead circuit at the terminals *a*-*b* in Fig. 4.23(a) must be equal to R_{Th} in Fig. 4.23(b) because the two circuits are equivalent. Thus, R_{Th} is the input resistance at the terminals when the independent sources are turned off, as shown in Fig. 4.24(b); that is,

$$R_{Th} = R_{in} \tag{4.7}$$

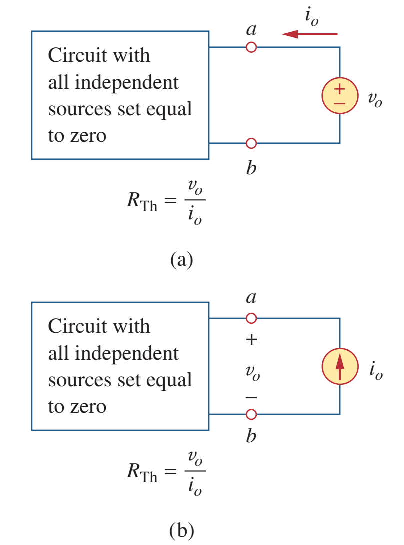


Figure 4.25
Finding R_{Th} when circuit has dependent sources.

Later we will see that an alternative way of finding R_{Th} is $R_{Th} = v_{oc}/i_{sc}$.

To apply this idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

■ **CASE 1** If the network has no dependent sources, we turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b, as shown in Fig. 4.24(b).

■ **CASE 2** If the network has dependent sources, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables. We apply a voltage source v_o at terminals a and b and determine the resulting current i_o. Then $R_{Th} = v_o/i_o$, as shown in Fig. 4.25(a). Alternatively, we may insert a current source i_o at terminals a-b as shown in Fig. 4.25(b) and find the terminal voltage v_o. Again $R_{Th} = v_o/i_o$. Either of the two approaches will give the same result. In either approach we may assume any value of v_o and i_o. For example, we may use $v_o = 1$ V or $i_o = 1$ A, or even use unspecified values of v_o or i_o.

It often occurs that R_{Th} takes a negative value. In this case, the negative resistance ($v = -iR$) implies that the circuit is supplying power. This is possible in a circuit with dependent sources; Example 4.10 will illustrate this.

Thevenin's theorem is very important in circuit analysis. It helps simplify a circuit. A large circuit may be replaced by a single independent voltage source and a single resistor. This replacement technique is a powerful tool in circuit design.

As mentioned earlier, a linear circuit with a variable load can be replaced by the Thevenin equivalent, exclusive of the load. The equivalent network behaves the same way externally as the original circuit. Consider a linear circuit terminated by a load R_L, as shown in Fig. 4.26(a). The current I_L through the load and the voltage V_L across the load are easily determined once the Thevenin equivalent of the circuit at the load's terminals is obtained, as shown in Fig. 4.26(b). From Fig. 4.26(b), we obtain

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} \tag{4.8a}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th} \tag{4.8b}$$

Note from Fig. 4.26(b) that the Thevenin equivalent is a simple voltage divider, yielding V_L by mere inspection.

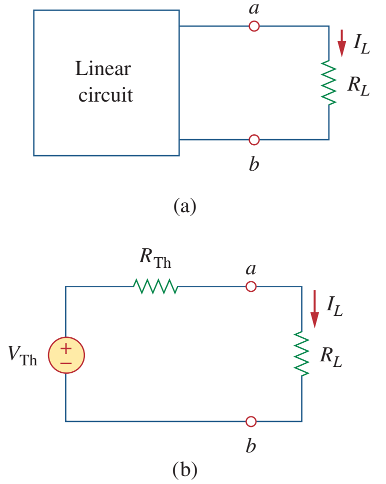


Figure 4.26
A circuit with a load: (a) original circuit, (b) Thevenin equivalent.

Example 4.8

Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals a-b. Then find the current through $R_L = 6$, 16, and 36 Ω.

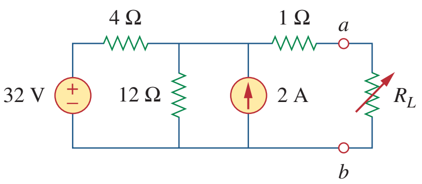


Figure 4.27
For Example 4.8.

Solution:

We find R_{Th} by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

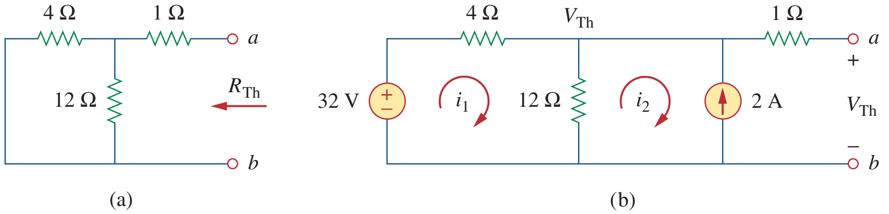


Figure 4.28
For Example 4.8: (a) finding R_{Th}, (b) finding V_{Th}.

To find V_{Th}, consider the circuit in Fig. 4.28(b). Applying mesh analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\text{ A}$$

Solving for i_1, we get $i_1 = 0.5$ A. Thus,

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\text{ V}$$

Alternatively, it is even easier to use nodal analysis. We ignore the 1-Ω resistor since no current flows through it. At the top node, KCL gives

$$\frac{32 - V_{Th}}{4} + 2 = \frac{V_{Th}}{12}$$

or

$$96 - 3V_{Th} + 24 = V_{Th} \quad \Rightarrow \quad V_{Th} = 30\text{ V}$$

as obtained before. We could also use source transformation to find V_{Th}.

The Thevenin equivalent circuit is shown in Fig. 4.29. The current through R_L is

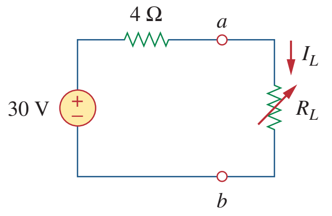


Figure 4.29
The Thevenin equivalent circuit for Example 4.8.

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3\text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5\text{ A}$$

When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75\text{ A}$$

Practice Problem 4.8

Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit of Fig. 4.30. Then find I.

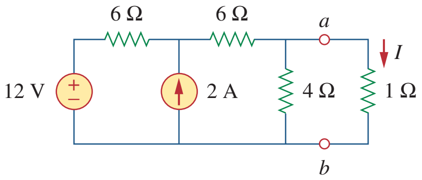


Figure 4.30
For Practice Prob. 4.8.

Answer: $V_{\text{Th}} = 6$ V, $R_{\text{Th}} = 3\ \Omega$, $I = 1.5$ A.

Example 4.9

Find the Thevenin equivalent of the circuit in Fig. 4.31 at terminals *a-b*.

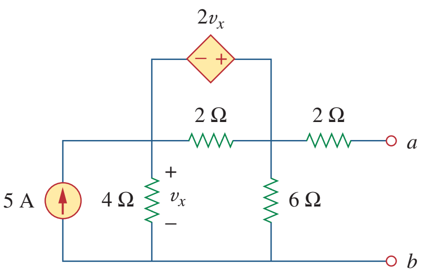


Figure 4.31
For Example 4.9.

Solution:
This circuit contains a dependent source, unlike the circuit in the previous example. To find R_{Th}, we set the independent source equal to zero but leave the dependent source alone. Because of the presence of the dependent source, however, we excite the network with a voltage source v_o connected to the terminals as indicated in Fig. 4.32(a). We may set $v_o = 1$ V to ease calculation, since the circuit is linear. Our goal is to find the current i_o through the terminals, and then obtain $R_{\text{Th}} = 1/i_o$. (Alternatively, we may insert a 1-A current source, find the corresponding voltage v_o, and obtain $R_{\text{Th}} = v_o/1$.)

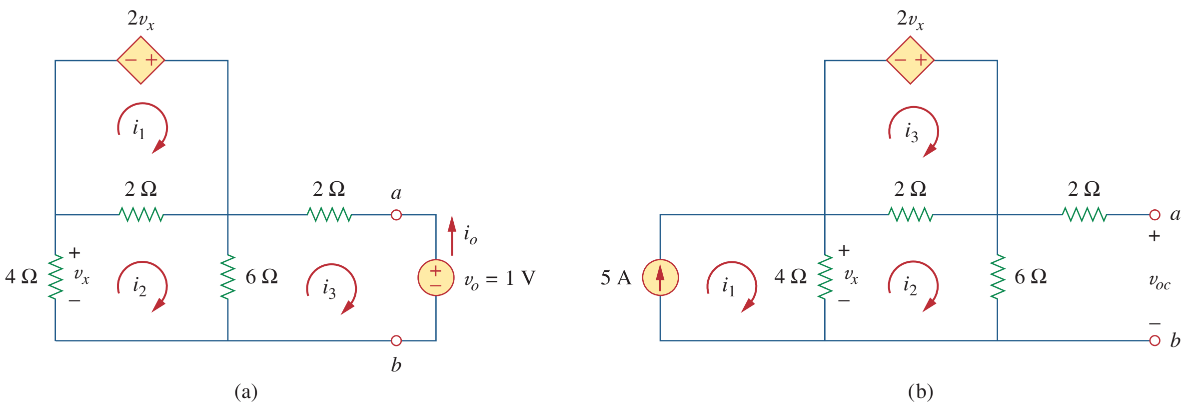


Figure 4.32
Finding R_{Th} and V_{Th} for Example 4.9.

Applying mesh analysis to loop 1 in the circuit of Fig. 4.32(a) results in

$$-2v_x + 2(i_1 - i_2) = 0 \qquad \text{or} \qquad v_x = i_1 - i_2$$

But $-4i_2 = v_x = i_1 - i_2$; hence,

$$i_1 = -3i_2 \tag{4.9.1}$$

For loops 2 and 3, applying KVL produces

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0 \tag{4.9.2}$$

$$6(i_3 - i_2) + 2i_3 + 1 = 0 \tag{4.9.3}$$

Solving these equations gives

$$i_3 = -\frac{1}{6}\text{ A}$$

But $i_o = -i_3 = 1/6$ A. Hence,

$$R_{\text{Th}} = \frac{1\text{ V}}{i_o} = 6\ \Omega$$

To get V_{Th}, we find v_{oc} in the circuit of Fig. 4.32(b). Applying mesh analysis, we get

$$i_1 = 5 \tag{4.9.4}$$

$$-2v_x + 2(i_3 - i_2) = 0 \quad \Rightarrow \quad v_x = i_3 - i_2 \tag{4.9.5}$$

$$4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$$

or

$$12i_2 - 4i_1 - 2i_3 = 0 \tag{4.9.6}$$

But $4(i_1 - i_2) = v_x$. Solving these equations leads to $i_2 = 10/3$. Hence,

$$V_{\text{Th}} = v_{oc} = 6i_2 = 20\text{ V}$$

The Thevenin equivalent is as shown in Fig. 4.33.

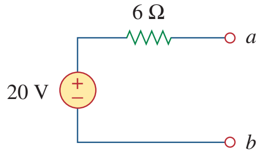


Figure 4.33
The Thevenin equivalent of the circuit in Fig. 4.31.

Practice Problem 4.9

Find the Thevenin equivalent circuit of the circuit in Fig. 4.34 to the left of the terminals.

Answer: $V_{\text{Th}} = 5.333$ V, $R_{\text{Th}} = 444.4$ mΩ.

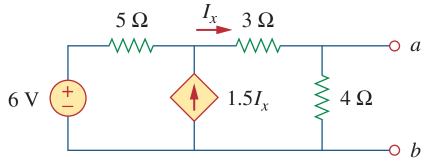


Figure 4.34
For Practice Prob. 4.9.

Example 4.10

Determine the Thevenin equivalent of the circuit in Fig. 4.35(a) at terminals *a-b*.

Solution:

1. **Define.** The problem is clearly defined; we are to determine the Thevenin equivalent of the circuit shown in Fig. 4.35(a).
2. **Present.** The circuit contains a 2-Ω resistor in parallel with a 4-Ω resistor. These are, in turn, in parallel with a dependent current source. It is important to note that there are no independent sources.
3. **Alternative.** The first thing to consider is that, since we have no independent sources in this circuit, we must excite the circuit externally. In addition, when you have no independent sources you will not have a value for V_{Th}; you will only have to find R_{Th}.

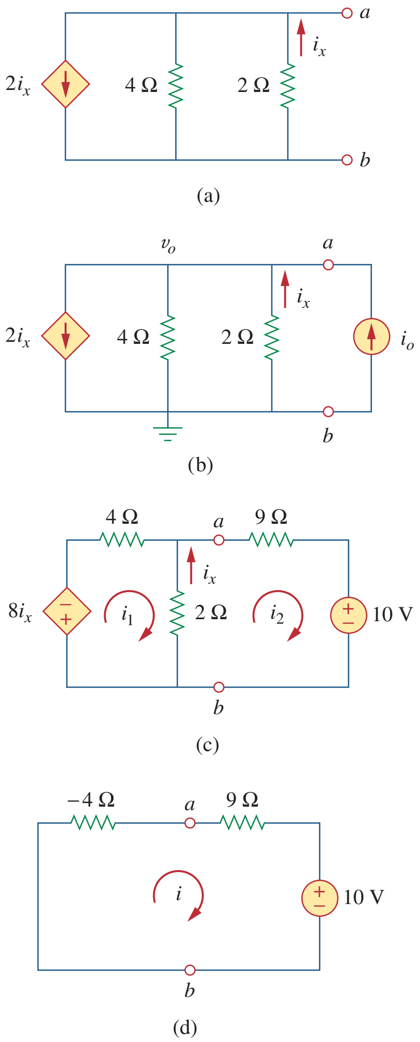


Figure 4.35
For Example 4.10.

The simplest approach is to excite the circuit with either a 1-V voltage source or a 1-A current source. Since we will end up with an equivalent resistance (either positive or negative), I prefer to use the current source and nodal analysis which will yield a voltage at the output terminals equal to the resistance (with 1 A flowing in, v_o is equal to 1 times the equivalent resistance).

As an alternative, the circuit could also be excited by a 1-V voltage source and mesh analysis could be used to find the equivalent resistance.

4. **Attempt.** We start by writing the nodal equation at a in Fig. 4.35(b) assuming $i_o = 1$ A.

$$2i_x + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0 \qquad \textbf{(4.10.1)}$$

Since we have two unknowns and only one equation, we will need a constraint equation.

$$i_x = (0 - v_o)/2 = -v_o/2 \qquad \textbf{(4.10.2)}$$

Substituting Eq. (4.10.2) into Eq. (4.10.1) yields

$$2(-v_o/2) + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0$$
$$= (-1 + \tfrac{1}{4} + \tfrac{1}{2})v_o - 1 \qquad \text{or} \qquad v_o = -4\text{ V}$$

Since $v_o = 1 \times R_{Th}$, then $R_{Th} = v_o/1 = \mathbf{-4\ \Omega}$.

The negative value of the resistance tells us that, according to the passive sign convention, the circuit in Fig. 4.35(a) is supplying power. Of course, the resistors in Fig. 4.35(a) cannot supply power (they absorb power); it is the dependent source that supplies the power. This is an example of how a dependent source and resistors could be used to simulate negative resistance.

5. **Evaluate.** First of all, we note that the answer has a negative value. We know this is not possible in a passive circuit, but in this circuit we do have an active device (the dependent current source). Thus, the equivalent circuit is essentially an active circuit that can supply power.

Now we must evaluate the solution. The best way to do this is to perform a check, using a different approach, and see if we obtain the same solution. Let us try connecting a 9-Ω resistor in series with a 10-V voltage source across the output terminals of the original circuit and then the Thevenin equivalent. To make the circuit easier to solve, we can take and change the parallel current source and 4-Ω resistor to a series voltage source and 4-Ω resistor by using source transformation. This, with the new load, gives us the circuit shown in Fig. 4.35(c).

We can now write two mesh equations.

$$8i_x + 4i_1 + 2(i_1 - i_2) = 0$$
$$2(i_2 - i_1) + 9i_2 + 10 = 0$$

Note, we only have two equations but have 3 unknowns, so we need a constraint equation. We can use

$$i_x = i_2 - i_1$$

This leads to a new equation for loop 1. Simplifying leads to

$$(4 + 2 - 8)i_1 + (-2 + 8)i_2 = 0$$

or

$$-2i_1 + 6i_2 = 0 \qquad \text{or} \qquad i_1 = 3i_2$$
$$-2i_1 + 11i_2 = -10$$

Substituting the first equation into the second gives

$$-6i_2 + 11i_2 = -10 \qquad \text{or} \qquad i_2 = -10/5 = \mathbf{-2\ A}$$

Using the Thevenin equivalent is quite easy since we have only one loop, as shown in Fig. 4.35(d).

$$-4i + 9i + 10 = 0 \qquad \text{or} \qquad i = -10/5 = \mathbf{-2\ A}$$

6. **Satisfactory?** Clearly we have found the value of the equivalent circuit as required by the problem statement. Checking does validate that solution (we compared the answer we obtained by using the equivalent circuit with one obtained by using the load with the original circuit). We can present all this as a solution to the problem.

Practice Problem 4.10

Obtain the Thevenin equivalent of the circuit in Fig. 4.36.

Answer: $V_{\text{Th}} = 0$ V, $R_{\text{Th}} = -7.5\ \Omega$.

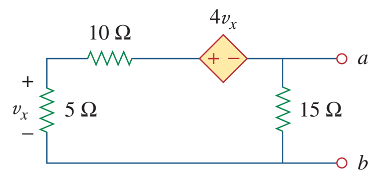


Figure 4.36
For Practice Prob. 4.10.

4.6 Norton's Theorem

In 1926, about 43 years after Thevenin published his theorem, E. L. Norton, an American engineer at Bell Telephone Laboratories, proposed a similar theorem.

> **Norton's theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Thus, the circuit in Fig. 4.37(a) can be replaced by the one in Fig. 4.37(b).

The proof of Norton's theorem will be given in the next section. For now, we are mainly concerned with how to get R_N and I_N. We find R_N in the same way we find R_{Th}. In fact, from what we know about source transformation, the Thevenin and Norton resistances are equal; that is,

$$\boxed{R_N = R_{\text{Th}}} \tag{4.9}$$

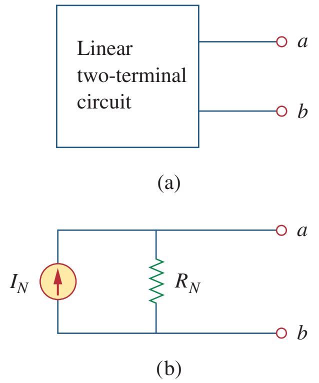


Figure 4.37
(a) Original circuit, (b) Norton equivalent circuit.

To find the Norton current I_N, we determine the short-circuit current flowing from terminal a to b in both circuits in Fig. 4.37. It is evident

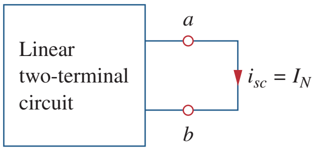


Figure 4.38
Finding Norton current I_N.

that the short-circuit current in Fig. 4.37(b) is I_N. This must be the same short-circuit current from terminal a to b in Fig. 4.37(a), since the two circuits are equivalent. Thus,

$$I_N = i_{sc} \tag{4.10}$$

shown in Fig. 4.38. Dependent and independent sources are treated the same way as in Thevenin's theorem.

Observe the close relationship between Norton's and Thevenin's theorems: $R_N = R_{Th}$ as in Eq. (4.9), and

$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}} \tag{4.11}$$

The Thevenin and Norton equivalent circuits are related by a source transformation.

This is essentially source transformation. For this reason, source transformation is often called Thevenin-Norton transformation.

Since V_{Th}, I_N, and R_{Th} are related according to Eq. (4.11), to determine the Thevenin or Norton equivalent circuit requires that we find:

- The open-circuit voltage v_{oc} across terminals a and b.
- The short-circuit current i_{sc} at terminals a and b.
- The equivalent or input resistance R_{in} at terminals a and b when all independent sources are turned off.

We can calculate any two of the three using the method that takes the least effort and use them to get the third using Ohm's law. Example 4.11 will illustrate this. Also, since

$$V_{Th} = v_{oc} \tag{4.12a}$$

$$I_N = i_{sc} \tag{4.12b}$$

$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N \tag{4.12c}$$

the open-circuit and short-circuit tests are sufficient to find any Thevenin or Norton equivalent, of a circuit which contains at least one independent source.

Example 4.11

Find the Norton equivalent circuit of the circuit in Fig. 4.39 at terminals a-b.

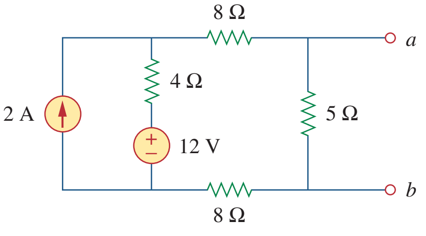


Figure 4.39
For Example 4.11.

Solution:
We find R_N in the same way we find R_{Th} in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find R_N. Thus,

$$R_N = 5 \,\|\, (8 + 4 + 8) = 5 \,\|\, 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

To find I_N, we short-circuit terminals a and b, as shown in Fig. 4.40(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2\text{ A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1\text{ A} = i_{sc} = I_N$$

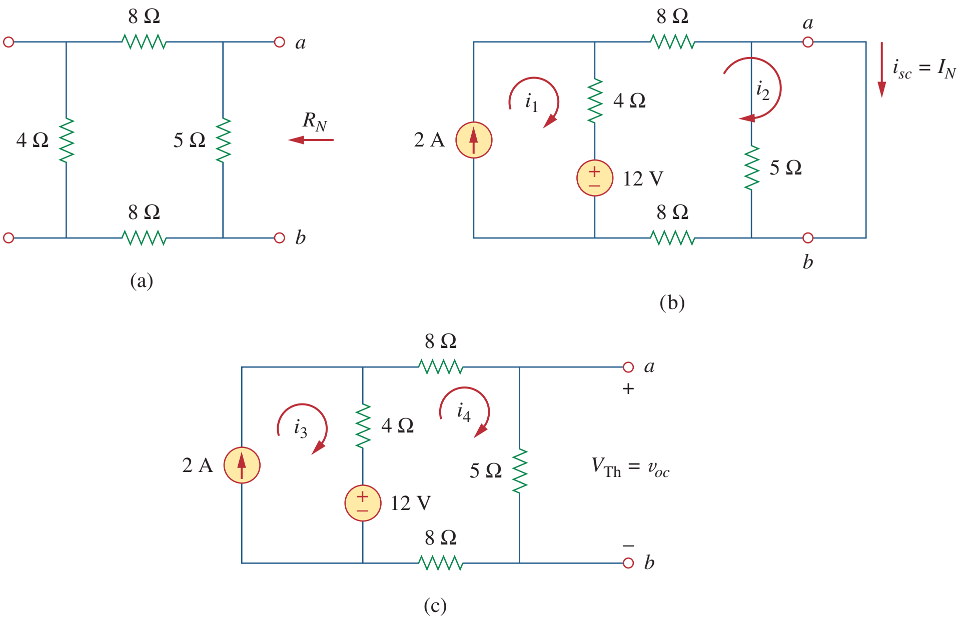


Figure 4.40
For Example 4.11; finding: (a) R_N, (b) $I_N = i_{sc}$, (c) $V_{Th} = v_{oc}$.

Alternatively, we may determine I_N from V_{Th}/R_{Th}. We obtain V_{Th} as the open-circuit voltage across terminals a and b in Fig. 4.40(c). Using mesh analysis, we obtain

$$i_3 = 2 \text{ A}$$

$$25i_4 - 4i_3 - 12 = 0 \quad \Rightarrow \quad i_4 = 0.8 \text{ A}$$

and

$$v_{oc} = V_{Th} = 5i_4 = 4 \text{ V}$$

Hence,

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1 \text{ A}$$

as obtained previously. This also serves to confirm Eq. (4.12c) that $R_{Th} = v_{oc}/i_{sc} = 4/1 = 4\ \Omega$. Thus, the Norton equivalent circuit is as shown in Fig. 4.41.

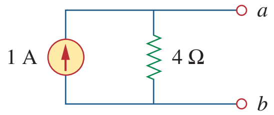


Figure 4.41
Norton equivalent of the circuit in Fig. 4.39.

Practice Problem 4.11

Find the Norton equivalent circuit for the circuit in Fig. 4.42, at terminals *a*-*b*.

Answer: $R_N = 3\ \Omega$, $I_N = 4.5$ A.

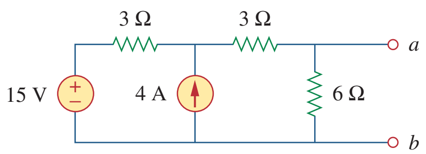


Figure 4.42
For Practice Prob. 4.11.

Example 4.12

Using Norton's theorem, find R_N and I_N of the circuit in Fig. 4.43 at terminals *a*-*b*.

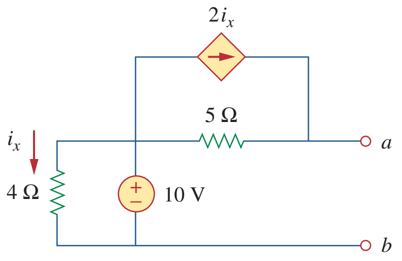


Figure 4.43
For Example 4.12.

Solution:

To find R_N, we set the independent voltage source equal to zero and connect a voltage source of $v_o = 1$ V (or any unspecified voltage v_o) to the terminals. We obtain the circuit in Fig. 4.44(a). We ignore the 4-Ω resistor because it is short-circuited. Also due to the short circuit, the 5-Ω resistor, the voltage source, and the dependent current source are all in parallel. Hence, $i_x = 0$. At node *a*, $i_o = \frac{1v}{5\Omega} = 0.2$ A, and

$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

To find I_N, we short-circuit terminals *a* and *b* and find the current i_{sc}, as indicated in Fig. 4.44(b). Note from this figure that the 4-Ω resistor, the 10-V voltage source, the 5-Ω resistor, and the dependent current source are all in parallel. Hence,

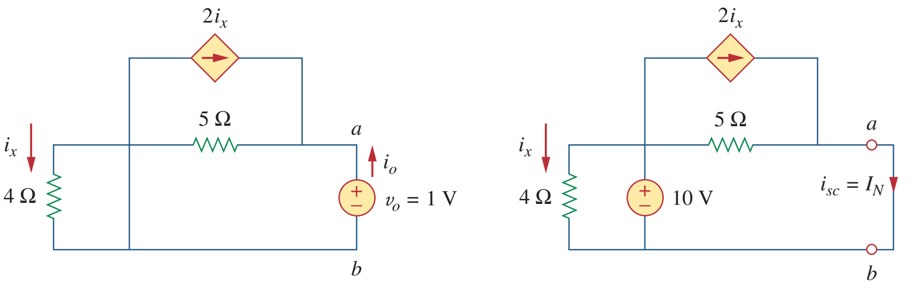
$$i_x = \frac{10}{4} = 2.5\ \text{A}$$

At node *a*, KCL gives

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\ \text{A}$$

Thus,

$$I_N = 7\ \text{A}$$



(a) (b)

Figure 4.44
For Example 4.12: (a) finding R_N, (b) finding I_N.

Practice Problem 4.12

Find the Norton equivalent circuit of the circuit in Fig. 4.45 at terminals *a*-*b*.

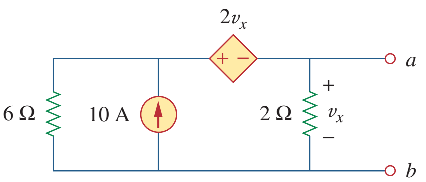


Figure 4.45
For Practice Prob. 4.12.

Answer: $R_N = 1\ \Omega$, $I_N = 10$ A.