2.4 Comparison of Measured and Accepted Values

Performing an experiment without drawing some sort of conclusion has little merit. A few experiments may have mainly qualitative results—the appearance of an interference pattern on a ripple tank or the color of light transmitted by some optical system—but the vast majority of experiments lead to *quantitative* conclusions, that is, to a statement of numerical results. It is important to recognize that the statement of *a single measured number is completely uninteresting.* Statements that the density

[3]This is not always so. For example, if you look up the refractive index of glass, you find values ranging from 1.5 to 1.9, depending on the composition of the glass. In an experiment to measure the refractive index of a piece of glass whose composition is unknown, the accepted value is therefore no more than a rough guide to the expected answer.

[4]Here is an example: If you measure the ratio of a circle's circumference to its diameter, the true answer is exactly π. (Obviously such an experiment is rather contrived.)

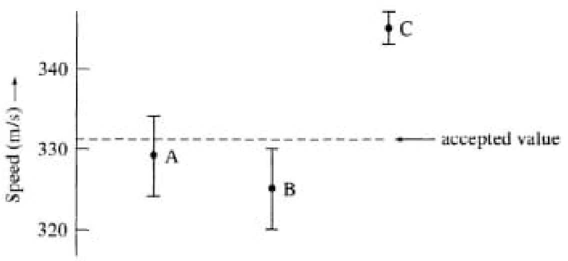

Figure 2.2. Three measurements of the speed of sound at standard temperature and pressure. Because the accepted value (331 m/s) is within Student A's margins of error, her result is satisfactory. The accepted value is just outside Student B's margin of error, but his measurement is nevertheless acceptable. The accepted value is *far outside* Student C's stated margins, and his measurement is definitely unsatisfactory.

of some metal was measured as 9.3 ± 0.2 gram/cm^3 or that the momentum of a cart was measured as 0.051 ± 0.004 kg·m/s are, by themselves, of no interest. An interesting conclusion must *compare two or more numbers*: a measurement with the accepted value, a measurement with a theoretically predicted value, or several measurements, to show that they are related to one another in accordance with some physical law. It is in such comparison of numbers that error analysis is so important. This and the next two sections discuss three typical experiments to illustrate how the estimated uncertainties are used to draw a conclusion.

Perhaps the simplest type of experiment is a measurement of a quantity whose accepted value is known. As discussed, this exercise is a somewhat artificial experiment peculiar to the teaching laboratory. The procedure is to measure the quantity, estimate the experimental uncertainty, and compare these values with the accepted value. Thus, in an experiment to measure the speed of sound in air (at standard temperature and pressure), Student A might arrive at the conclusion

$$\text{A's measured speed} = 329 \pm 5 \text{ m/s}, \qquad (2.12)$$

compared with the

$$\text{accepted speed} = 331 \text{ m/s}. \qquad (2.13)$$

Student A might choose to display this result graphically as in Figure 2.2. She should certainly include in her report both Equations (2.12) and (2.13) next to each other, so her readers can clearly appreciate her result. She should probably add an explicit statement that because the accepted value lies inside her margins of error, her measurement seems satisfactory.

The meaning of the uncertainty δx is that the correct value of x *probably* lies between $x_{best} - \delta x$ and $x_{best} + \delta x$; it is certainly *possible* that the correct value lies slightly outside this range. Therefore, a measurement can be regarded as satisfactory even if the accepted value lies slightly outside the estimated range of the measured value. For example, if Student B found the value

$$\text{B's measured speed} = 325 \pm 5 \text{ m/s},$$

he could certainly claim that his measurement is consistent with the accepted value of 331 m/s.

On the other hand, if the accepted value is well outside the margins of error (the discrepancy is appreciably more than twice the uncertainty, say), there is reason to think something has gone wrong. For example, suppose the unlucky Student C finds

$$\text{C's measured speed} = 345 \pm 2 \text{ m/s} \tag{2.14}$$

compared with the

$$\text{accepted speed} = 331 \text{ m/s}. \tag{2.15}$$

Student C's discrepancy is 14 m/s, which is seven times bigger than his stated uncertainty (see Figure 2.2). He will need to check his measurements and calculations to find out what has gone wrong.

Unfortunately, the tracing of C's mistake may be a tedious business because of the numerous possibilities. He may have made a mistake in the measurements or calculations that led to the answer 345 m/s. He may have estimated his uncertainty incorrectly. (The answer 345 ± 15 m/s would have been acceptable.) He also might be comparing his measurement with the wrong accepted value. For example, the accepted value 331 m/s is the speed of sound at standard temperature and pressure. Because standard temperature is 0°C, there is a good chance the measured speed in (2.14) was *not* taken at standard temperature. In fact, if the measurement was made at 20°C (that is, normal room temperature), the correct accepted value for the speed of sound is 343 m/s, and the measurement would be entirely acceptable.

Finally, and perhaps most likely, a discrepancy such as that between (2.14) and (2.15) may indicate some undetected source of systematic error (such as a clock that runs consistently slow, as discussed in Chapter 1). Detection of such systematic errors (ones that consistently push the result in one direction) requires careful checking of the calibration of all instruments and detailed review of all procedures.

2.5 Comparison of Two Measured Numbers

Many experiments involve measuring two numbers that theory predicts should be equal. For example, the law of conservation of momentum states that the total momentum of an isolated system is constant. To test it, we might perform a series of experiments with two carts that collide as they move along a frictionless track. We could measure the total momentum of the two carts before (p) and after (q) they collide and check whether $p = q$ within experimental uncertainties. For a single pair of measurements, our results could be

$$\text{initial momentum } p = 1.49 \pm 0.03 \text{ kg·m/s}$$

and

$$\text{final momentum } q = 1.56 \pm 0.06 \text{ kg·m/s}.$$

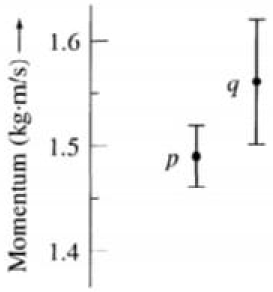

Figure 2.3. Measured values of the total momentum of two carts before (p) and after (q) a collision. Because the margins of error for p and q overlap, these measurements are certainly consistent with conservation of momentum (which implies that p and q should be equal).

Here, the range in which p probably lies (1.46 to 1.52) *overlaps* the range in which q probably lies (1.50 to 1.62). (See Figure 2.3.) Therefore, these measurements are consistent with conservation of momentum. If, on the other hand, the two probable ranges were not even close to overlapping, the measurements would be inconsistent with conservation of momentum, and we would have to check for mistakes in our measurements or calculations, for possible systematic errors, and for the possibility that some external forces (such as gravity or friction) are causing the momentum of the system to change.

If we repeat similar pairs of measurements several times, what is the best way to display our results? First, using a table to record a sequence of similar measurements is usually better than listing the results as several distinct statements. Second, the uncertainties often differ little from one measurement to the next. For example, we might convince ourselves that the uncertainties in all measurements of the initial momentum p are about $\delta p \approx 0.03$ kg·m/s and that the uncertainties in the final q are all about $\delta q \approx 0.06$ kg·m/s. If so, a good way to display our measurements would be as shown in Table 2.1.

Table 2.1. Measured momenta (kg·m/s).

Trial number	Initial momentum p (all ± 0.03)	Final momentum q (all ± 0.06)
1	1.49	1.56
2	3.10	3.12
3	2.16	2.05
etc.		

For each pair of measurements, the probable range of values for p overlaps (or nearly overlaps) the range of values for q. If this overlap continues for all measurements, our results can be pronounced consistent with conservation of momentum. Note that our experiment does not *prove* conservation of momentum; no experiment can. The best you can hope for is to conduct many more trials with progressively smaller uncertainties and that all the results are *consistent* with conservation of momentum.

In a real experiment, Table 2.1 might contain a dozen or more entries, and checking that each final momentum q is consistent with the corresponding initial momentum p could be tedious. A better way to display the results would be to add a fourth column that lists the differences $p - q$. If momentum is conserved, these values should be consistent with zero. The only difficulty with this method is that we must now compute the uncertainty in the difference $p - q$. This computation is performed as follows. Suppose we have made measurements

$$(\text{measured } p) = p_{\text{best}} \pm \delta p$$

and

$$(\text{measured } q) = q_{\text{best}} \pm \delta q.$$

The numbers p_{best} and q_{best} are our best estimates for p and q. Therefore, the best estimate for the difference $(p - q)$ is $(p_{\text{best}} - q_{\text{best}})$. To find the uncertainty in $(p - q)$, we must decide on the highest and lowest probable values of $(p - q)$. The highest value for $(p - q)$ would result if p had its *largest* probable value, $p_{\text{best}} + \delta p$, at the same time that q had its *smallest* value $q_{\text{best}} - \delta q$. Thus, the highest probable value for $p - q$ is

$$\text{highest probable value} = (p_{\text{best}} - q_{\text{best}}) + (\delta p + \delta q). \tag{2.16}$$

Similarly, the lowest probable value arises when p is smallest $(p_{\text{best}} - \delta p)$, but q is largest $(q_{\text{best}} + \delta q)$. Thus,

$$\text{lowest probable value} = (p_{\text{best}} - q_{\text{best}}) - (\delta p + \delta q). \tag{2.17}$$

Combining Equations (2.16) and (2.17), we see that the *uncertainty in the difference* $(p - q)$ is the ***sum*** $\delta p + \delta q$ *of the original uncertainties*. For example, if

$$p = 1.49 \pm 0.03 \text{ kg·m/s}$$

and

$$q = 1.56 \pm 0.06 \text{ kg·m/s},$$

then

$$p - q = -0.07 \pm 0.09 \text{ kg·m/s}.$$

We can now add an extra column for $p - q$ to Table 2.1 and arrive at Table 2.2.

Table 2.2. Measured momenta (kg·m/s).

Trial number	Initial p (all ± 0.03)	Final q (all ± 0.06)	Difference $p - q$ (all ± 0.09)
1	1.49	1.56	−0.07
2	3.10	3.12	−0.02
3	2.16	2.05	0.11
etc.			

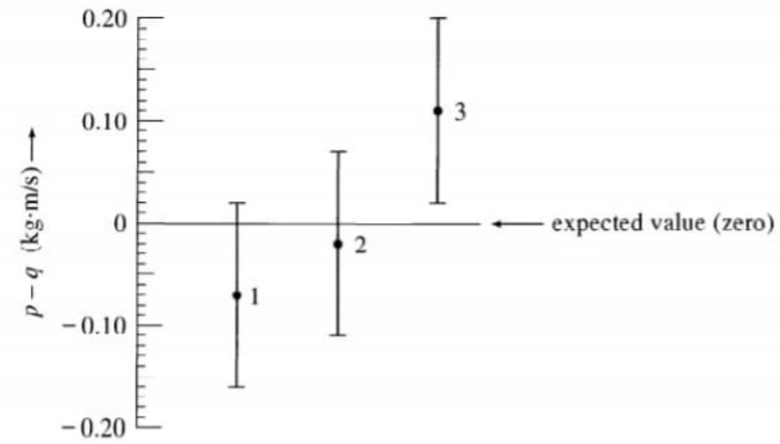

Figure 2.4. Three trials in a test of the conservation of momentum. The student has measured the total momentum of two carts before and after they collide (p and q, respectively). If momentum is conserved, the differences $p - q$ should all be zero. The plot shows the value of $p - q$ with its error bar for each trial. The expected value 0 is inside the margins of error in trials 1 and 2 and only slightly outside in trial 3. Therefore, these results are consistent with the conservation of momentum.

Whether our results are consistent with conservation of momentum can now be seen at a glance by checking whether the numbers in the final column are consistent with zero (that is, are less than, or comparable with, the uncertainty 0.09). Alternatively, and perhaps even better, we could plot the results as in Figure 2.4 and check visually. Yet another way to achieve the same effect would be to calculate the *ratios* q/p, which should all be consistent with the expected value $q/p = 1$. (Here, we would need to calculate the uncertainty in q/p, a problem discussed in Chapter 3.)

Our discussion of the uncertainty in $p - q$ applies to the difference of any two measured numbers. If we had measured any two numbers x and y and used our measured values to compute the difference $x - y$, by the argument just given, the resulting uncertainty in the difference would be the *sum* of the separate uncertainties in x and y. We have, therefore, established the following provisional rule:

Uncertainty in a Difference
(Provisional Rule)

If two quantities x and y are measured with uncertainties δx and δy, and if the measured values x and y are used to calculate the difference $q = x - y$, the *uncertainty in* q is the *sum of the uncertainties in* x *and* y:

$$\delta q \approx \delta x + \delta y. \tag{2.18}$$

2.9 Multiplying Two Measured Numbers

Perhaps the greatest importance of fractional errors emerges when we start multiplying measured numbers by each other. For example, to find the momentum of a body, we might measure its mass m and its velocity v and then multiply them to give the momentum $p = mv$. Both m and v are subject to uncertainties, which we will have to estimate. The problem, then, is to find the uncertainty in p that results from the known uncertainties in m and v.

First, for convenience, let us rewrite the standard form

$$(\text{measured value of } x) = x_{\text{best}} \pm \delta x$$

in terms of the fractional uncertainty, as

$$(\text{measured value of } x) = x_{\text{best}}\left(1 \pm \frac{\delta x}{|x_{\text{best}}|}\right). \tag{2.23}$$

For example, if the fractional uncertainty is 3%, we see from (2.23) that

$$(\text{measured value of } x) = x_{\text{best}}\left(1 \pm \frac{3}{100}\right);$$

that is, 3% uncertainty means that x probably lies between x_{best} times 0.97 and x_{best} times 1.03,

$$(0.97) \times x_{\text{best}} \leqslant x \leqslant (1.03) \times x_{\text{best}}.$$

We will find this a useful way to think about a measured number that we will have to multiply.

Let us now return to our problem of calculating $p = mv$, when m and v have been measured, as

$$(\text{measured } m) = m_{\text{best}}\left(1 \pm \frac{\delta m}{|m_{\text{best}}|}\right) \tag{2.24}$$

and

$$(\text{measured } v) = v_{\text{best}}\left(1 \pm \frac{\delta v}{|v_{\text{best}}|}\right) \tag{2.25}$$

Because m_{best} and v_{best} are our best estimates for m and v, our best estimate for $p = mv$ is

$$(\text{best estimate for } p) = p_{\text{best}} = m_{\text{best}} v_{\text{best}}.$$

The largest probable values of m and v are given by (2.24) and (2.25) with the plus signs. Thus, the largest probable value for $p = mv$ is

$$(\text{largest value for } p) = m_{\text{best}} v_{\text{best}} \left(1 + \frac{\delta m}{|m_{\text{best}}|}\right)\left(1 + \frac{\delta v}{|v_{\text{best}}|}\right). \quad (2.26)$$

The smallest probable value for p is given by a similar expression with two minus signs. Now, the product of the parentheses in (2.26) can be multiplied out as

$$\left(1 + \frac{\delta m}{|m_{\text{best}}|}\right)\left(1 + \frac{\delta v}{|v_{\text{best}}|}\right) = 1 + \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|} + \frac{\delta m}{|m_{\text{best}}|}\frac{\delta v}{|v_{\text{best}}|}. \quad (2.27)$$

Because the two fractional uncertainties $\delta m/|m_{\text{best}}|$ and $\delta v/|v_{\text{best}}|$ are small numbers (a few percent, perhaps), their product is extremely small. Therefore, the last term in (2.27) can be neglected. Returning to (2.26), we find

$$(\text{largest value of } p) = m_{\text{best}} v_{\text{best}} \left(1 + \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}\right).$$

The smallest probable value is given by a similar expression with two minus signs. Our measurements of m and v, therefore, lead to a value of $p = mv$ given by

$$(\text{value of } p) = m_{\text{best}} v_{\text{best}} \left(1 \pm \left[\frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}\right]\right).$$

Comparing this equation with the general form

$$(\text{value of } p) = p_{\text{best}} \left(1 \pm \frac{\delta p}{|p_{\text{best}}|}\right),$$

we see that the best estimate for p is $p_{\text{best}} = m_{\text{best}} v_{\text{best}}$ (as we already knew) and that the *fractional uncertainty in p is the sum of the fractional uncertainties in m and v,*

$$\frac{\delta p}{|p_{\text{best}}|} = \frac{\delta m}{|m_{\text{best}}|} + \frac{\delta v}{|v_{\text{best}}|}.$$

If, for example, we had the following measurements for m and v,

$$m = 0.53 \pm 0.01 \text{ kg}$$

and

$$v = 9.1 \pm 0.3 \text{ m/s},$$

the best estimate for $p = mv$ is

$$p_{\text{best}} = m_{\text{best}} v_{\text{best}} = (0.53) \times (9.1) = 4.82 \text{ kg·m/s}.$$

To compute the uncertainty in p, we would first compute the fractional errors

$$\frac{\delta m}{m_{\text{best}}} = \frac{0.01}{0.53} = 0.02 = 2\%$$

and

$$\frac{\delta v}{v_{\text{best}}} = \frac{0.3}{9.1} = 0.03 = 3\%.$$

The fractional uncertainty in p is then the sum:

$$\frac{\delta p}{p_{\text{best}}} = 2\% + 3\% = 5\%.$$

If we want to know the absolute uncertainty in p, we must multiply by p_{best}:

$$\delta p = \frac{\delta p}{p_{\text{best}}} \times p_{\text{best}} = 0.05 \times 4.82 = 0.241.$$

We then round δp and p_{best} to give us our final answer

$$(\text{value of } p) = 4.8 \pm 0.2 \text{ kg}\cdot\text{m/s}.$$

The preceding considerations apply to any product of two measured quantities. We have therefore discovered our second general rule for the propagation of errors. If we measure any two quantities x and y and form their product, the uncertainties in the original two quantities "propagate" to cause an uncertainty in their product. This uncertainty is given by the following rule:

Uncertainty in a Product
(Provisional Rule)

If two quantities x and y have been measured with small fractional uncertainties $\delta x/|x_{\text{best}}|$ and $\delta y/|y_{\text{best}}|$, and if the measured values of x and y are used to calculate the product $q = xy$, then the *fractional uncertainty in q is the sum of the fractional uncertainties in x and y,*

$$\frac{\delta q}{|q_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta y}{|y_{\text{best}}|}. \tag{2.28}$$

I call this rule "provisional," because, just as with the rule for uncertainty in a difference, I will replace it with a more precise rule later on. Two other features of this rule also need to be emphasized. First, the derivation of (2.28) required that the fractional uncertainties in x and y both be small enough that we could neglect their product. This requirement is almost always true in practice, and I will always assume it. Nevertheless, remember that if the fractional uncertainties are *not* much smaller than 1, the rule (2.28) may not apply. Second, even when x and y have different dimensions, (2.28) balances dimensionally because all fractional uncertainties are dimensionless.