CHAPTER 2: Fractional Uncertainties

Absolute vs Fractional Uncertainties

We have seen that the correct reporting of a physical measurement requires that one write the “best” value with a quoted error uncertainty. In general then, we write

$$\text{Measured } x = x_{\text{best}} \pm \delta x$$

In this case δx is the absolute uncertainty of the measurement. However, it is often more clear to write the *fractional uncertainty* of the measurement instead of the absolute uncertainty. The idea is that a measurement with a relatively large fractional uncertainty is not as meaningful as a measurement with a relatively small fractional uncertainty.

Definition of Fractional Uncertainty

The fractional uncertainty is just the ratio of the absolute uncertainty, δx to the best value x_{best}:

$$\text{Fractional Uncertainty } \equiv \frac{\delta x}{x_{\text{best}}}$$

In general, the absolute uncertainty δx will be numerically less than the measured best value x_{best}. Otherwise the measurement is generally not worth reporting. The only exception to this rule is in the case where one is trying to make a so-called “null measurement”. In that case only the absolute uncertainty has meaning.

For all non-null measurements which have their absolute uncertainties less than the measured quantity itself, then it is standard practice to quote the fractional uncertainty as a percentage. For example, suppose one measures a length l as 50 cm with an uncertainty of 1 cm. Then the absolute quote is

$$l = 50 \pm 1 \text{ cm}$$

while the fractional uncertainty is

$$\text{Fractional Uncertainty } = \frac{\delta l}{l} = \frac{1}{50} = 0.02$$

So the result can also be given as

$$l = 50 \text{ cm} \pm 2\%$$

Propagation of Errors

Introduction to Propagation of Errors

In determining a physical quantity it is only very rarely that we make a direct experimental measurement on the quantity itself. Much more often it is the case that we make direct measurements on quantities which are mathematically related to the unknown physical quantity. Then by a series of either simple or complicated mathematical steps, we arrive at the unknown quantity of interest. So we can think of *directly measured* physical quantities and *indirectly measured* physical quantities. The directly measured physical quantities will have errors associated with them as we discussed in the opening lectures, which errors reflect the measurement apparatus or techniques used. The indirect, or calculated physical quantities will also have errors associated, and these errors will be the *propagated* errors from the direct measurement errors. We have already seen in Chapter 2 the first examples of error propagation involving addition or subtraction and multiplication or division. The book's rules are:

In adding or subtracting physical quantities, the **absolute** *measurement errors of the individual quantities are added to obtain the absolute error in the calculated quantity.*

In multiplying or dividing physical quantities, the **fractional** *measurements errors of the individual quantities are added to obtain the fractional error in the calculated quantity.*

The "professional physicists" rules are much the same except that we add the squares of the individual errors and take the square roots of that sum.

Uncertainties in Direct Measurements

We have seen that the error quotes for direct measurements are generally associated with how well can one read the measurement device. The classic example is a meter stick calibrated in millimeter gradations where a half millimeter error quote would be quite reasonable. However, one may have a digital device, such as a digital stop watch, which is capable of giving readings in the milli-seconds. You might then be tempted to quote time errors to ± 0.001 second. However, this would probably be a mistake in most introductory mechanics labs. The actual physical process itself, such as the fall time in a gravity experiment or an Atwoods machine, is likely to vary by more than a milli-second even under very controlled conditions. Hence a quote of ± 0.05 seconds would be more realistic. Thus it is important to distinguish between the precision of the measuring device and the precision associated with the measurement itself.

Uncertainties in Direct Measurements

Counting Experiments

A very common type of physical measurement is simple a "counting experiment". The typical example is the decay of a long-lived (years) radioactive source for which the emission of particles is completely random over a very short time interval (say milli-seconds), but has a definite average rate over a longer time interval (say minutes).

In such experiments, one counts the emitted particles for a fixed time such as a few minutes and records the number of counts. That number, divided by the time interval, constitutes the count rate. One may count again for the same amount of time and find a slightly different number of counts with a slightly different count rate. Here we see an example of a *directly measured* physical quantity (the counts) and a derived physical quantity (the count rate).

The question which arises is what is the error associated with this kind of counting experiment. The answer is remarkably simple. The error associated with a counting measurement is simply the square root of the number of counts. This is the prime example of *statistical error*. Specifically, counting experiments are part of what is know as *Poisson Distribution* which is fully discussed in Chapter 11.

Statisical Error in Counting Experiments and Count Rate Errors

To recapitulate the discussion above for counting experiments, if one has an experiment where N counts are measured, then the uncertainty δN in that measurement is given by the Poisson Statistics formula:

$$\delta N = \sqrt{N}$$

So, to take an easy number, say we measure 100 counts in a 2 minute period. Then we can say that the error in that measurement is $\sqrt{100} = 10$, and we can then quote

$$N = 100 \pm 10$$

However, one will typically not quote the actual number of counts, but rather the rate of counts in a given time period. Let's give the rate quantity the symbol R, so obviously in this simple example

$$R = \frac{N}{T} = \frac{100}{2} = 50 \text{ counts/minute}$$

Now what is the error associated with the rate quantity R?

Error Propagation Examples

Count Rate Errors and Error Propagation

The answer to the question of the error in a count rate is an example of error propagation. Here we have a directly measured physical quantity, counts, divided by a fixed constant, time (2 minutes), in order to obtain a derived quantity. In such a case, the absolute error in the derived quantity is the absolute error in the measured quantity divided by the fixed constant.

So in the case at error

$$R = \frac{N}{T} \Longrightarrow \delta R = \frac{\delta N}{T} = \frac{\sqrt{N}}{T}$$

In the present example, then $\delta R = \sqrt{100}/2 = 5$ counts/minute. So we would quote the result as $R = 50 \pm 5$ counts per minute.

First General Rule for Error Propagation of Calculated Quantities The book (page 5) gives the first general rule according to the following formula. If there is a quantity q which is calculated as the product of a constant B and a measured quantity x

$$q = Bx$$

and the measured quantity has an error δx, then the error δq in the calculated quantity is given as

$$\delta q = B\delta x$$

In our count rate example above we actually had $B = 1/T$ and were dividing by a fixed quantity instead of multiplying. But mathematically, it makes no difference.

Another way of stating this same rule is that the fractional error in the derived quantity is the same as the fractional error in the directly measured quantity.

Power Law and an Error Propagation

A second general rule about error propagation applies to a power law dependence. Take for example $q = x^n$ where n may or may not be an integer. Then the error δq is given as

$$\frac{\delta q}{q} = n\frac{\delta x}{x}$$

If n is an integer, you can think of this as adding up n times the fractional error in x since q is the product of x taken n times.

Propagation of Errors with One Variable

Arbitrary Function of One Variable

Suppose we have a calculated physical quantity q which depends upon a measured physical quantity x according to the general function

$$q = q(x)$$

In error analysis, we want to know that how much uncertainty δq do we attach to q when the uncertainty in x is given as δx
The answer to this question comes directly from calculus. You should have seen in elementary calculus that it is always possible to expand a well-behaved function about some point in terms of increasing orders of derivatives

$$q(x) = q(x_0) + (x - x_0)\frac{dq}{dx}_{x=x_0} + (x - x_0)^2\frac{d^2q}{dx^2}_{x=x_0} + \ldots$$

So if we think of $\delta x = x - x_0$ as the uncertainty about the true value of x, then the uncertainty $\delta q = q(x) - q(x_0)$, and we have

$$\delta q = \delta x\frac{dq}{dx}_{x=x_0} + (\delta x)^2\frac{d^2q}{dx^2}_{x=x_0} + \ldots$$

Now we make our usual assumption that δx is small (and also that the higher order derivatives are not large, which means "well-behaved"), and obtaining

$$\delta q \approx \delta x\frac{dq}{dx}_{x=x_0}$$

Of course, we only care about the absolute values of all the quantities so it is fair to write

$$\delta q \approx \delta x\left|\frac{dq}{dx}\right|$$

and we are assuming that we are evaluating the derivative at some given measured point x_0.

Uncertainty for One Variable

Example of Uncertainty for One Variable

We have derived the general rule for a function of one variable that the uncertainty in the calculated variable is given by

$$\delta q \approx \delta x |\frac{dq}{dx}|$$

As an example (page 65 suppose we take x to be the angle θ and q to be the function $\cos\theta$. Then we have the uncertainty in q is given by

$$q = \cos\theta \Longrightarrow \delta q = \delta\theta(\frac{d\cos\theta}{d\theta}) = \delta\theta \sin\theta$$

where, as usual in calculus, all angular quantities are expressed in radians.
To be specific, supposed we measured $\theta = 20 \pm 3$ degrees. Then we have $q = \cos 20 = 0.94$. As for the error δq in q we evaluate

$$\delta q = \delta\theta \sin\theta = (0.05)(0.34) = 0.02$$

Again, remember that we have to convert the angular quantity $\delta\theta$ 3 degrees into radians So the value for q is quoted as

$$q = 0.94 \pm 0.02$$

The Power Law Re-Visited

Previously we showed that for power law functions, $q = x^n$ then the fractional uncertainty in q was n times the uncertainty in x

$$\frac{\delta q}{q} = n\frac{\delta x}{x}$$

You can think of this as a product function of x by n times, with all the errors adding up correlated.
This same result can be derived using our general rule for functions of one variable. We have

$$q = x^n \Longrightarrow \delta q = n(\delta x)x^{n-1}$$

Now divide both sides by the original expression $q = x^n$

$$\frac{\delta q}{q} = \frac{n(\delta x)x^{n-1}}{x^n} = n\frac{\delta x}{x}$$

Uncertainty with Two Variables

The Pendulum Example

The pendulum experiment is a good example of a calculated quantity, the acceleration due to gravity g, depending upon two measured quantities, a length l and a time T. As you know

$$T = 2\pi\sqrt{\frac{l}{g}}$$

which we can re-write as a calculated g quantity

$$g = \frac{4\pi^2 l}{T^2}$$

We assume that we measure l with an uncertainty δl and T with and uncertainty δT. So, what is the uncertainty in the calculated value of g?
We view this calculated quantity as a product of a constant $(4\pi^2)$ which as no uncertainty, a linear term l, and a quadratic term T^2. The quantities l and T are measured independently of one another, and there should be no correlation in their respective uncertainties. Hence the fractional error in g is obtained by adding the fractional errors of the two terms in quadrature. The fractional error in the l term is simply $\delta l/l$, and the fractional error in the T^2 term is by the power rule equal to $2\delta T/T$. We thus obtain

$$\frac{\delta g}{g} = \sqrt{(\frac{\delta l}{l})^2 + (2\frac{\delta T}{T})^2}$$

Calculated Pendulum Example

We can now put specific numbers in to show what happens. We take as an exmaple (page 68) the following measurements and quoted errors

$$l = 92.95 \pm 0.1 \text{ cm}$$

$$T = 1.936 \pm 0.004 \text{ sec}$$

We obtain directly from these numbers that $g = 979$ cm/s^2, and that

$$\frac{\delta g}{g} = \sqrt{(\frac{0.1}{92.95}^2 + (2\frac{0.004}{1.936}^2)} = 0.004$$

$$\Longrightarrow \delta g = .004 \times g = 4 \text{ cm/s}^2$$

So we make the experimental result as $g = 979 \pm 4$ cm/s^2.

Uncertainty with Multiple Variables

General Formula for Multiple Variables

The extension of this error analysis to multiple variables is very straightforward. Suppose we take a quantity q to be a function of several independently measured variables as follows

$$q = q(x, \ldots, z)$$

Each of these independently measured variables $(x, \ldots, z)$ has its own, independent error uncertainty $(\delta x, \ldots, \delta z)$. Then we can used the calculus of more than one variable to arrive at the following result

$$\delta q \approx \sqrt{(\frac{\partial q}{\partial x}\delta x)^2 + \ldots + (\frac{\partial q}{\partial z}\delta z)^2}$$

The approximation sign indicates that we are assuming that the individual errors are relatively small, and that the function itself is equivalently well behaved in the region where all the physical quantities $(x, \ldots, z)$ are being measured.

The partial derivative, $\partial q/\partial x$, means take the derivative of the q function with respect to x and assume all the other variables are constant. And so on for $\partial q/\partial y, \ldots, \partial q/\partial z$.

This is a very powerful error analysis formula which will work in all cases: one variable, two variables, and more than two variables. You should memorize this formula. Morever, with the help of *Mathematica*, you can very quickly get the error uncertainties of even complicated, multi-variable functions. This is because *Mathematica* provides a means of evaluation derivatives for you analytically.