Example 14.7. Assume that a population consists of 7 similar containers having the following weights (kilograms):

9.8, 10.2, 10.4, 9.8, 10.0, 10.2 and 9.6.

(a) Find the mean μ and the standard deviation σ of the given population.

(b) Draw random samples of 2 containers without replacement and calculate the mean weight $\bar{X}$ of each sample.

(c) Form a frequency distribution of $\bar{X}$ and a sampling distribution of $\bar{X}$.

(d) Find the mean and the standard deviation of the sampling distribution of $\bar{X}$.

(a) The population mean μ and standard deviation σ are

$$\mu = \frac{\Sigma X}{N} = \frac{9.8 + 10.2 + \ldots + 9.6}{7} = \frac{70.0}{7} = 10.0 \text{ kg; and}$$

$$\sigma = \sqrt{\frac{\Sigma(X-\mu)^2}{N}} = \sqrt{\frac{(9.8-10)^2+(10.2-10)^2+\ldots+(9.6-10)^2}{7}}$$

$$= \sqrt{\frac{0.48}{7}} = \sqrt{0.0686} = 0.262 \text{ kg.}$$

(b) Let the containers be identified as A, B, C, D, E, F and G. Now the number of possible random samples of $n = 2$ containers without replacement is $\binom{7}{2} = 21$. The 21 possible random samples with the values of their mean weights are given on page 32:

(c) The frequency distribution of $\bar{X}$ and the sampling distribution of the mean $\bar{X}$, which is just the relative frequency distribution of $\bar{X}$, are obtained below:

(i) Frequency Distribution of $\bar{X}$

Sample Mean $\bar{x}$	Tally	f
9.7	II	2
9.8	II	2
9.9	IIII	4
10.0	~~IIII~~	5
10.1	IIII	4
10.2	II	2
10.3	II	2
Σ	--	21

(ii) Sampling Distribution of $\bar{X}$

Sample Mean $\bar{x}$	Probability $f(\bar{x})$
9.7	2/21
9.8	2/21
9.9	4/21
10.0	5/21
10.1	4/21
10.2	2/21
10.3	2/21
Σ	1

Sample No.	Sample Combination	Weights in X_1	Samples X_2	Sample Mean weight ($\bar{X}$)
1	A, B	9.8,	10.2	10.0
2	A, C	9.8,	10.4	10.1
3	A, D	9.8,	9.8	9.8
4	A, E	9.8,	10.0	9.9
5	A, F	9.8,	10.2	10.0
6	A, G	9.8,	9.6	9.7
7	B, C	10.2,	10.4	10.3
8	B, D	10.2,	9.8	10.0
9	B, E	10.2,	10.0	10.1
10	B, F	10.2,	10.2	10.2
11	B, G	10.2,	9.6	9.9
12	C, D	10.4,	9.8	10.1
13	C, E	10.4,	10.0	10.2
14	C, F	10.4,	10.2	10.3
15	C, G	10.4,	9.6	10.0
16	D, E	9.8,	10.0	9.9
17	D, F	9.8,	10.2	10.0
18	D, G	9.8,	9.6	9.7
19	E, F	10.0,	10.2	10.1
20	E, G	10.0,	9.6	9.8
21	F, G	10.2,	9.6	9.9

(d) The mean and standard deviation of sampling distribution of $\bar{X}$, are computed below:

Sample Mean $\bar{x}$	Probability $f(\bar{x})$	$\bar{x}\,f(\bar{x})$	$\bar{x}-\mu_{\bar{x}}$	$(\bar{x}-\mu_{\bar{x}})^2$	$(\bar{x}-\mu_{\bar{x}})^2 f(\bar{x})$
9.7	2/21	19.4/21	−0.3	0.09	0.18/21
9.8	2/21	19.6/21	−0.2	0.04	0.08/21
9.9	4/21	39.6/21	−0.1	0.01	0.04/21
10.0	5/21	50.0/21	0	0	0
10.1	4/21	40.4/21	+0.1	0.01	0.04/21
10.2	2/21	20.4/21	0.2	0.04	0.08/21
10.3	2/21	20.6/21	0.3	0.09	0.18/21
Σ	1	10.0	--	--	0.6/21

$$\mu_{\bar{x}} = \sum \bar{x}\, f(\bar{x}) = 10.0\ kg, \text{ and}$$

$$\sigma_{\bar{x}} = \sqrt{\sum (\bar{x}-\mu_{\bar{x}})^2 f(\bar{x})} = \sqrt{\frac{0.6}{21}} = \sqrt{0.0286} = 0.17\ kg,$$

which is a smaller value indicating that the sampling distribution of the mean is more concentrated about the population mean.

Example 14.8. A sample of size $n=3$ is to be randomly selected without replacement from a population that has $N=5$ items whose values are 0, 3, 6, 9 and 12.

(a) Find the sampling distribution of the sample mean, $\bar{X}$.

(b) Calculate the mean and the standard deviation of $\bar{X}$, and verify that

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1}.$$

Let the items be designated by the letters A, B, C, D and E.

(a) The number of samples of size $n=3$ that could be drawn without replacement from a population of size $N=5$ is

$$\binom{5}{3} = \frac{5!}{2!\,3!} = 10.$$

The 10 possible samples and their means are given below:

Sample No.	Sample Combinations	Sample Values	Sample Mean ($\bar{X}$)
1	*A, B, C*	0, 3, 6	3
2	*A, B, D*	0, 3, 9	4
3	*A, B, E*	0, 3, 12	5
4	*A, C, D*	0, 6, 9	5
5	*A, C, E*	0, 6, 12	6
6	*A, D, E*	0, 9, 12	7
7	*B, C, D*	3, 6, 9	6
8	*B, C, E*	3, 6, 12	7
9	*B, D, E*	3, 9, 12	8
10	*C, D, E*	6, 9, 12	9

The sampling distribution is obtained by listing all possible me and their probabilities (relative frequencies) as below:

Sampling Distribution of $\bar{X}$

Sample Mean $\bar{X}$	Number of sample means (f)	Probability $f(\bar{x})$
3	1	1/10
4	1	1/10
5	2	2/10
6	2	2/10
7	2	2/10
8	1	1/10
9	1	1/10
Σ	10	1

(b) Next, we calculate the mean and the standard deviation (th *standard error*) of the sampling distribution of the mean as follows:

Calculation of Mean and S.D. of Sampling Distribution of $\bar{X}$.

Sample Mean $\bar{x}$	Probability $f(\bar{x})$	$\bar{x}\, f(\bar{x})$	$\bar{x}^2 f(\bar{x})$
3	1/10	3/10	9/10
4	1/10	4/10	16/10
5	2/10	10/10	50/10
6	2/10	12/10	72/10
7	2/10	14/10	98/10
8	1/10	8/10	64/10
9	1/10	9/10	81/10
$\sum$	1	60/10	390/10

Now $\mu_{\bar{x}} = \sum \bar{x}\, f(\bar{x}) = \dfrac{60}{10} = 6$, and

$$\sigma_{\bar{x}} = \sqrt{[\sum \bar{x}^2 f(\bar{x})] - [\sum \bar{x} f(\bar{x})]^2}$$

$$= \sqrt{\frac{390}{10} - \left(\frac{60}{10}\right)^2} = \sqrt{39 - 36} = \sqrt{3} = 1.732$$

In order to verify the given result, we first calculate the mean μ and the variance σ^2 of the given population. Thus

$$\mu = \frac{1}{5}[0 + 3 + 6 + 9 + 12] = \frac{1}{5}[30] = 6, \text{ and}$$

$$\sigma^2 = \frac{1}{5}[(0-6)^2 + (3-6)^2 + (6-6)^2 + (9-6)^2 + (12-6)^2] = 18$$

Verification: Now $\dfrac{\sigma^2}{n} \cdot \dfrac{N-n}{N-1} = \dfrac{18}{3} \cdot \dfrac{5-3}{5-1} = \dfrac{18 \times 2}{3 \times 4} = 3 = \sigma^2_{\bar{x}}$

Hence the result.