4.4.8 Cauchy's Equation of Motion

Newton's second law of motion can be stated as

$$\sum \mathbf{F} = M\mathbf{a} \tag{4.92}$$

where the left-hand side is the sum of all external forces acting on the control volume and consists solely of surface and body forces:

$$\sum \mathbf{F} = \sum \mathbf{F}_s + \sum \mathbf{F}_b \tag{4.93}$$

The body forces $\mathbf{F}_b$ act on the center of mass of the control volume. Examples are gravitational, buoyant, and electromagnetic forces. The gravitational body force, $\mathbf{G}$, on the fluid volume $\forall$ is

$$\mathbf{G} = \int_{\forall} \rho \mathbf{g} \, d\forall \tag{4.94}$$

where $\mathbf{g}$ is the acceleration due to gravity. Since the x, y axes lie in a horizontal plane relative to the surface of the earth, assuming the earth is a homogeneous sphere, we can state

$$\mathbf{g} = -32.2\,\mathbf{k}\ (\text{ft/s}^2) = -9.81\,\mathbf{k}\ (\text{m/s}^2) \tag{4.95}$$

$\sum F_s + \sum F_b = Ma$

$\int_A \sigma \cdot dA + \int_V \rho g \, dV = \int_V \rho a \, dV$

Using the relationship between mass M and density ρ and expressing the surface force in terms of the stress dyadic $\mathbf{P}$, we obtain

$F = Ma$

$$\int_{\forall} (\mathbf{g} - \mathbf{a}) \rho \, d\forall + \int_{A} \mathbf{P} \cdot d\mathbf{A} = 0 \tag{4.96}$$

The area integral can be expressed in terms of a volume integral through Gauss's theorem, Eq. (B.37), such that Eq. (4.96) becomes

$$\int_{\forall} [(\mathbf{g} - \mathbf{a}) \rho + \nabla \cdot \mathbf{P}] \, d\forall = 0 \tag{4.97}$$

Since the volume $\forall$ is completely arbitrary and the integrand is continuous throughout the volume, the integrand of Eq. (4.97) must also be zero:

$$\boxed{(\mathbf{g} - \mathbf{a}) \rho + \nabla \cdot \mathbf{P} = 0} \tag{4.98}$$

The above vector differential equation is called *Cauchy's equation of motion*. Physically, it represents a unique balance of body and surface forces with the inertial force. If the body forces do not nullify the surface forces a nonzero inertial force will cause the fluid to accelerate.

It is also important to state that Cauchy's equation of motion is valid for *both* incompressible and compressible fluids. The difference between these two fluids lies in the expressions for the stress. For an

- Incompressible fluid flow:

$$\mathbf{P} = 2\mu \dot{\mathbf{S}} - p\mathbf{I} \tag{4.59}$$

and for a

- Compressible fluid flow:

$$\mathbf{P} = 2\mu \dot{\mathbf{S}} - (p + \tfrac{2}{3}\mu \nabla \cdot \mathbf{V}) \mathbf{I} \tag{4.60}$$

The equation of motion as given by Eq. (4.98) can easily be expressed in Cartesian form, by use of Eqs. (1.45)–(1.47) for the acceleration vector $\mathbf{a}$, and Eq. (4.51) for the stress dyadic $\mathbf{P}$ to yield

1. x-component

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = g_x + \frac{1}{\rho} \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \right) \tag{4.99}$$

2. y-component

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y + \frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\right) \tag{4.100}$$

3. z-component

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z + \frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\right) \tag{4.101}$$

(recalling that $\mathbf{i} \cdot (\mathbf{ii}) = \mathbf{i}$, $\mathbf{i} \cdot (\mathbf{ij}) = \mathbf{j}$, $\mathbf{i} \cdot (\mathbf{ik}) = \mathbf{k}$ and any other dot product of $\mathbf{i}$ with the other unit dyads is zero).

A similar procedure is followed using cylindrical coordinates. The three scalar forms of Cauchy's equation of motion in cylindrical coordinates are

1. r-component

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + w\frac{\partial v_r}{\partial z} = g_r + \frac{1}{\rho}\left(\frac{1}{r}\frac{\partial (rp_{rr})}{\partial r} + \frac{1}{r}\frac{\partial p_{r\theta}}{\partial \theta} - \frac{p_{\theta\theta}}{r} + \frac{\partial p_{rz}}{\partial z}\right) \tag{4.102}$$

2. θ-component

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + w\frac{\partial v_\theta}{\partial z} = g_\theta + \frac{1}{\rho}\left[\frac{1}{r^2}\frac{\partial (r^2 p_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{\partial p_{\theta z}}{\partial z}\right] \tag{4.103}$$

3. z-component

$$\frac{\partial w}{\partial t} + v_r\frac{\partial w}{\partial r} + \frac{v_\theta}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = g_z + \frac{1}{\rho}\left[\frac{1}{r}\frac{\partial (rp_{rz})}{\partial r} + \frac{1}{r}\frac{\partial (p_{\theta z})}{\partial \theta} + \frac{\partial p_{zz}}{\partial z}\right] \tag{4.104}$$

If the fluid is inviscid, all the shear stresses vanish because $\mu = 0$, resulting in the stress dyadic

$$\mathbf{P} = \mathbf{ii}p_{xx} + \mathbf{jj}p_{yy} + \mathbf{kk}p_{zz}$$
$$= \mathbf{e}_r\mathbf{e}_r p_{rr} + \mathbf{e}_\theta\mathbf{e}_\theta p_{\theta\theta} + \mathbf{kk}p_{zz}$$

Thus only normal stresses remain, resulting in

$$\nabla \cdot \mathbf{P} = -\nabla p \tag{4.105}$$

for an inviscid fluid.

One counts 13 dependent variables in the above equations: three velocity components, three components of the body force per unit mass, the density, and six stresses. We evaluate the dependent variables in terms of four independent variables, x, y, z, and t (or r, θ, z, and t). Thirteen dependent variables require 13 equations.

For an *incompressible* fluid, the density is constant. For a spherical potential field, the body force per unit mass $\mathbf{g}$ is known and is given by Eq. (4.95). Thus, the number of unknowns is reduced to nine. Section 4.4.5 discusses the stress dyadic $\mathbf{P}$ with rate of strain resulting in six additional equations: Eqs. (4.61)–(4.66). However, we also pick up an additional dependent variable: pressure. Thus, we have 10 unknowns and 10 equations, sufficient to solve most incompressible flow problems.

For a compressible fluid, the density is a variable, so we have 11 unknowns and 10 equations, and we need an additional equation, supplied by the fluid's equation of state.

The next section consolidates everything so that for the incompressible case, the number of unknowns is reduced to four, with four equations. The compressible case is also given. Three of these equations are called the Navier-Stokes equations. The other equations have already been developed. One is the continuity equation, and the other is the equation of state.

Example 4.9

Consider a one-dimensional flow that is incompressible, steady, and inviscid. Calculate the velocity of the flow in terms of the pressure given the pressure $p = \gamma x$ at $z = 0$, and $u = 0$ at $x = 0$.

Solution:

Step 1.

Identify the characteristics of the fluid and flow field.

The fluid is incompressible and inviscid; the flow is one-dimensional and steady.

Step 2.

Write the appropriate form of the governing equations of flow

Example 4.9 *(Con't.)*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial D}{\partial x}\right) \quad (4.99)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial D}{\partial y}\right) \quad (4.100)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial D}{\partial z}\right) \quad (4.101)$$

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x} \quad (4.61)$$

$$p_{yy} = -p + 2\mu\frac{\partial v}{\partial y} \quad (4.62)$$

$$p_{zz} = -p + 2\mu\frac{\partial w}{\partial z} \quad (4.63)$$

$$p_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad (4.64)$$

$$p_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (4.65)$$

$$p_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \quad (4.66)$$

For inviscid steady one-dimensional flow, Eq. (4.99) in conjunction with Eqs. (4.61)–(4.66) becomes

$$u\frac{du}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{(i)}$$

since $\partial p/\partial y = 0$ and $\partial p/\partial z = -\rho g$. Integrating the result of Eq. (4.101) gives

$$p = -\rho g z + f(x) \quad \text{(ii)}$$

The function $f(x)$ is evaluated from the boundary condition $p = \gamma x$ at $z = 0$ with the result

$$p = -\rho g z + \gamma x \quad \text{(iii)}$$

Example 4.9 *(Con't.)*

Substituting Eq. (iii) into Eq. (i), then integrating, results in

$$u = \sqrt{2gx} \tag{iv}$$

We have just illustrated one way to obtain the velocity of the flow. Notice we had to carefully define what the fluid was, what the constraints were imposed on the flow, and what was taking place on the boundaries.

This completes the solution.

Example 4.10

Consider a steady inviscid two-dimensional flow of a fluid. Calculate the magnitude of the velocity $|\mathbf{V}|$ of the flow in terms of the pressure p and potential energy Ω, *given* that the slope of the path of the fluid particle is

$$\frac{dy}{dx} = \frac{v}{u} \tag{i}$$

and the gravitational acceleration **g** is the negative of the gradient of the potential energy

$$\mathbf{g} = -\nabla\Omega \tag{ii}$$

Solution:

The steps in the problem solution are identical to Example 4.9 except that this flow is now two-dimensional.

For inviscid steady two-dimensional flows, Eqs. (4.99) and (4.100) become with the help of Eqs. (4.61)–(4.66)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{iii}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{iv}$$

Substituting Eq. (ii) into Eqs. (iii) and (iv) results in

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{v}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \Omega}{\partial g} - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{vi}$$

Since the slope of the path of the fluid particle is given by Eq. (i), we substitute $v = u\dfrac{dy}{dx}$ into Eq. (v) and $u = v(dx/dy)$ into Eq. (vi) to obtain

Example 4.10 *(Con't.)*

$$u\frac{\partial u}{\partial x} + u\frac{dy}{dx}\frac{\partial u}{\partial y} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(vii)}$$

$$v\frac{dx}{dy}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(viii)}$$

Multiply Eq. (vii) by dx and Eq. (viii) by dy and add the resultant two equations:

$$u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + v\frac{\partial v}{\partial x}dx + v\frac{\partial v}{\partial y}dy$$

$$= -\left(\frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy\right) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right) \qquad \text{(ix)}$$

Since the flow is two-dimensional, $u = u(x, y)$, $v = v(x, y)$, $\Omega = \Omega(x, y)$, and $p = p(x, y)$ so that Eq. (ix) can be expressed in terms of exact differentials:

$$\frac{1}{2}d(u^2) + \frac{1}{2}d(v^2) = -d\Omega - \frac{1}{\rho}dp \qquad \text{(x)}$$

or it can be expressed in terms of the nabla operator ∇:

$$\nabla\left(\frac{p}{\rho} + \frac{V^2}{2} + \Omega\right) = 0 \qquad \text{(xi)}$$

Integrating Eq. (x) yields

$$\frac{1}{2}(u^2 + v^2) + \Omega + \frac{p}{\rho} = \text{const.} \qquad \text{(xii)}$$

Since the magnitude of the velocity **V** is defined as

$$|\mathbf{V}| = \sqrt{u^2 + v^2} \qquad \text{(xiii)}$$

we obtain

$$\boxed{|\mathbf{V}| = \sqrt{2}\left\{\text{const} - \frac{p}{\rho} - \Omega\right\}^{1/2}} \qquad \text{(xiv)}$$

This is the famous Bernoulli equation [Eq. (4.126)] which we shall apply to many different problems in Chap. 5.

This completes the solution.

Example 4.11

Oil is slowly flowing down the flat surface of a large rectangular tank because of gravity, as shown in Fig. E4.11. Its free-surface is exposed to air at 1 at-

Example 4.11 *(Con't.)*

mosphere. It has a thickness h. Let air be inviscid. If the flow of oil is steady and two-dimensional, determine (a) the stress **P** in terms of specific weight γ, thickness of oil h, and distance y, and (b) the velocity field **V**. (c) Show that the flow is rotational.

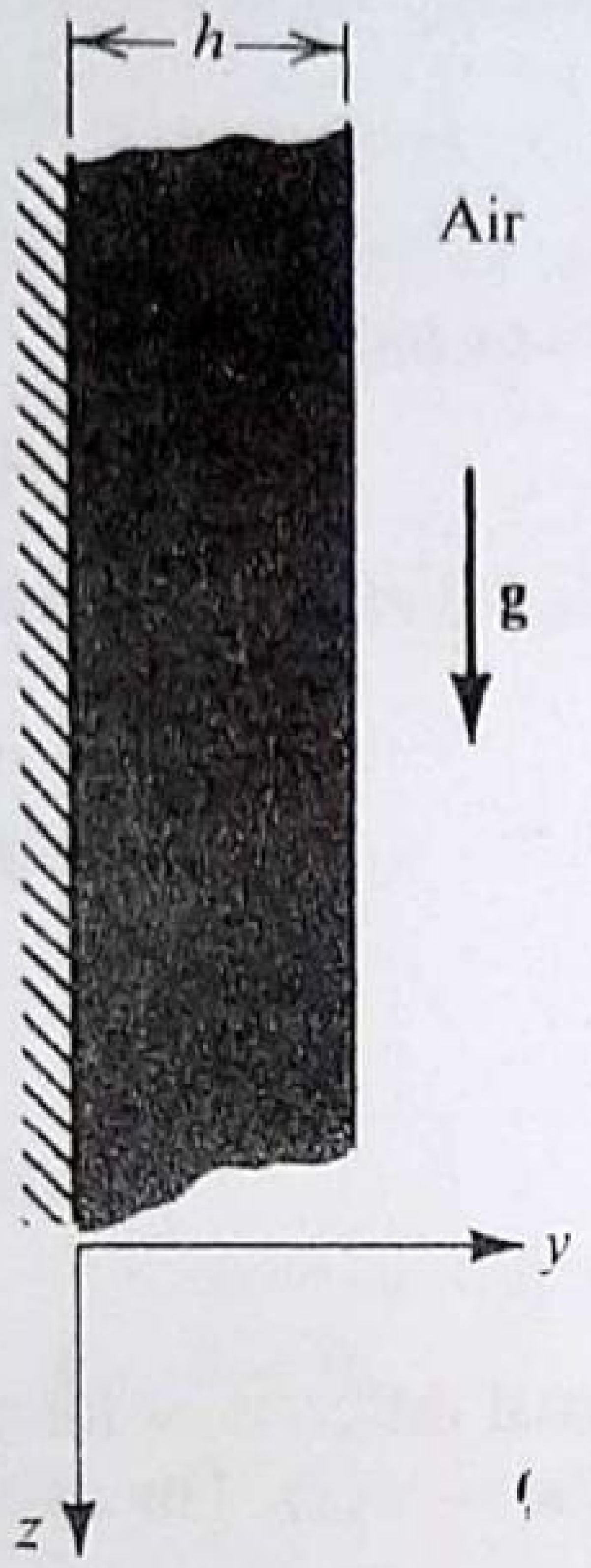

Figure E4.11

Solution:

Step 1.

Identify the characteristics of the fluid and flow field.

Oil is incompressible and viscous. The flow is steady and two-dimensional. The velocity components u, v are zero. Thus the velocity of the oil is in the z-direction and is expressed as

$$w = w(y, z) \tag{i}$$

Step 2.

Write the appropriate form of the governing equations.

From the conservation of mass, the continuity equation becomes

$$\overset{0}{\cancel{\frac{\partial u}{\partial x}}} + \overset{0}{\cancel{\frac{\partial v}{\partial y}}} + \frac{\partial w}{\partial z} = 0 \tag{ii}$$

Cauchy's equations of motion result in

$$\frac{Du}{Dt} = g_x + \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right) \tag{iii}$$

Example 4.11 *(Con't.)*

$$\frac{Dv}{Dt} = g_y + \frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}\right) \qquad \text{(iv)}$$

$$w\frac{\partial w}{\partial z} = g + \frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}\right) \qquad \text{(v)}$$

The result shown in Eq. (ii) states that $w \neq w(z)$, such that Eq. (i) becomes

$$w = w(y) \qquad \text{(vi)}$$

Since Cauchy's equations of motion are in terms of stresses, let us examine each separately. Since $u = v = 0$ and $w = w(y)$, Eqs. (4.61)–(4.66) reduce to

$$p_{xx} = p_{yy} = p_{zz} = -p = \text{const.} \qquad \text{(vii)}$$

Note, so long as the oil has a free-surface, there will be no pressure gradient; i.e., $\nabla p = 0$. In addition to Eq. (vii), we have

$$p_{xy} = p_{xz} = 0 \qquad \text{(viii)}$$

and

$$p_{yz} = \mu\frac{dw}{dy} \qquad \text{(ix)}$$

It is important to point out that we have gone from the partial differential form of Eq. (4.66) to the total differential form of Eq. (ix) since $w = w(y)$. The only finite equation among Eqs. (iii)–(v) is Eq. (v):

$$0 = g + \frac{1}{\rho}\left(\frac{\partial p_{yz}}{\partial y}\right) \qquad \text{(x)}$$

Step 3.
Solve the differential equation. Since $w = w(y)$, the partial differential equation of Eq. (x) is transformed to a total differential. Integrating Eq. (x) yields

$$p_{yz} = -\gamma y + c_1 \qquad \text{(xi)}$$

At $y = h$, the shear stress must vanish due to the stress boundary condition. Thus the stress becomes

$$p_{yz} = \gamma(h - y) \qquad \text{(xii)}$$

a) The stress tensor $\mathbf{P}$ can now be evaluated using Eq. (4.51) and Eqs. (vii), (viii), and (xii):

$$\mathbf{P} = -p\mathbf{ii} - p\mathbf{jj} - p\mathbf{kk} + \gamma(h - y)\mathbf{jk} \qquad \text{(xiii)}$$

Knowing the stress tensor enables us to evaluate such valuable engineering quantities as drag and normal forces on objects exposed to fluid flows (see Example 4.8).

Example 4.11 *(Con't.)*

(b) The velocity field **V** is obtained knowing the velocity components u, v, w. Since $u = v = 0$, we need only w. Equating the results of our shear stress p_{yz} [expressed by Eq. (xii) to Eq. (ix)] we integrate the result to obtain

$$w = \frac{\gamma}{\mu}(hy - y^2/2) + c_2 \tag{xiv}$$

To evaluate the constant of integration c_2, we use the kinematic boundary condition of no slip; i.e., at $y = 0$, $w = 0$. Hence, the velocity field is

$$\mathbf{V} = \frac{\gamma y}{\mu}(h - y/2)\mathbf{k} \tag{xv}$$

(c) To show the flow is rotational, we seek to evaluate the curl of the velocity vector **V**:

$$\nabla \times \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i}$$

$$= \frac{\gamma}{\mu}(h - y)\mathbf{i} \tag{xvi}$$

We discover that the vorticity component ζ_x is proportional to the shear stress p_{yz}, that is,

$$\zeta_x = \frac{p_{yz}}{\mu} \tag{xvii}$$

Thus if shear stresses exist in a flow we can expect fluid parcel rotations.

This completes the solution.