14.3 *Prandtl's Boundary Layer Equations*

Let us now discuss the approximate equations of motion for a two-dimensional laminar boundary layer. We make this approximation by estimating the order of magnitude of each term in the Navier-Stokes equations. The result will give us the Prandtl boundary layer equations, first published by Prandtl [1.29] in 1904. The following assumptions are made:

1. The motion is two-dimensional, and lies in a horizontal plane with the y-axis normal to the plate as shown in Fig. 14.3: thus gravitational effects are neglected.
2. The flow is laminar within the boundary layer.
3. The boundary layer thickness is much smaller than the length of the plate L, i.e., $\delta^* = \delta/L \ll 1$.
4. The flow is steady.

We start the discussion with the x and y components of the dimensionless Navier-Stokes equations as given by Eq. (7.26). We recall that our dimensionless independent variables are $x^* = x/L$, $y^* = y/L$, and $z^* = z/L$, and are based on a characteristic length L that is constant. Our dimensionless dependent variables are $u^* = u/U$, $v^* = v/U$, $w^* = w/U$, and $p^* = p/\rho U^2$, and are based on a characteristic velocity U

that is constant. Using the above dimensionless variables, we reduce the dimensionless Navier-Stokes equation (7.26) to

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{14.5}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{R}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{14.6}$$

and the dimensionless continuity equation is

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{14.7}$$

Use of Eq. (14.1) reveals that the orders of magnitude of $\partial u^*/\partial x^*$ and $\partial^2 u^*/\partial x^{*2}$ are 1, which means that $\partial v^*/\partial y^*$ must have an order of magnitude of 1 in order to satisfy continuity. Now, since v^* equals zero on the boundary, then $v^* = \int_0^{\delta^*} 1\, dy = \delta^*$. Thus the dimensionless y-component velocity has an order of magnitude δ^*. Also $\partial v^*/\partial x^*$ and $\partial^2 v^*/\partial x^{*2}$ have orders of magnitude δ^*. From Eq. (14.2) $\partial u^*/\partial y^*$ and $\partial^2 u^*/\partial y^{*2}$ have orders of magnitude of $1/\delta^*$ and $1/\delta^{*2}$, respectively. The dimensional pressure gradient $\partial p/\partial x$ is assumed to be known in advance from Bernoulli's equation applied to the outer inviscid flow

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho U \frac{dU}{dx} \tag{14.8}$$

The distribution of $U(x)$ along a surface is known from the inviscid analysis described in Chap. 12. Thus $\partial p^*/\partial x^*$ is retained since its order of magnitude is 1.

If we insert these orders of magnitude into Eqs. (14.5), (14.6), and (14.7), we obtain

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{14.9}$$

$$\frac{\partial p^*}{\partial y^*} = 0(\delta^*) \cong 0 \tag{14.10}$$

Equation (14.10) states that the pressure across the boundary layer does not change. The pressure is impressed on the boundary layer, and its value is determined by hydrodynamic considerations. This is all true only if the flow does not separate, and it will not separate if the flow is past a flat plate with no wall transpiration. Transforming back to the dimensional variables (u, v, p; x, y), we obtain the Prandtl boundary layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (14.11)^*$$

$$\frac{\partial p}{\partial y} = 0 \qquad (14.12)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (14.13)$$

subject to the following necessary and sufficient boundary conditions:

- No slip at the wall: $u = v = 0$ at $y = 0$ (14.14a)
- Patching: $u \to U$ as $y \to \delta$ (14.14b)

We solve the Prandtl boundary layer equations for $u(x, y)$ and $v(x, y)$ with $U(x)$ known from the outer inviscid flow analysis. The equations are solved by starting at the leading edge of the body and moving downstream to the separation point.**

Note that the remaining momentum Eq. (14.11) is still nonlinear. However, it does allow the no-slip boundary condition to be satisfied which constitutes a significant improvement over potential flow analysis in the solution of real fluid flow problems. The Prandtl boundary layer equations are thus a simplification of the Navier-Stokes equations. They can be regarded as *asymptotic equations of the Navier-Stokes equations in the limit of vanishing viscosity*.

Example 14.1

Using the Prandtl boundary layer Eq. (14.11), show that the velocity profile for a laminar flow past a flat plate has an infinite radius of curvature on the surface of the plate.

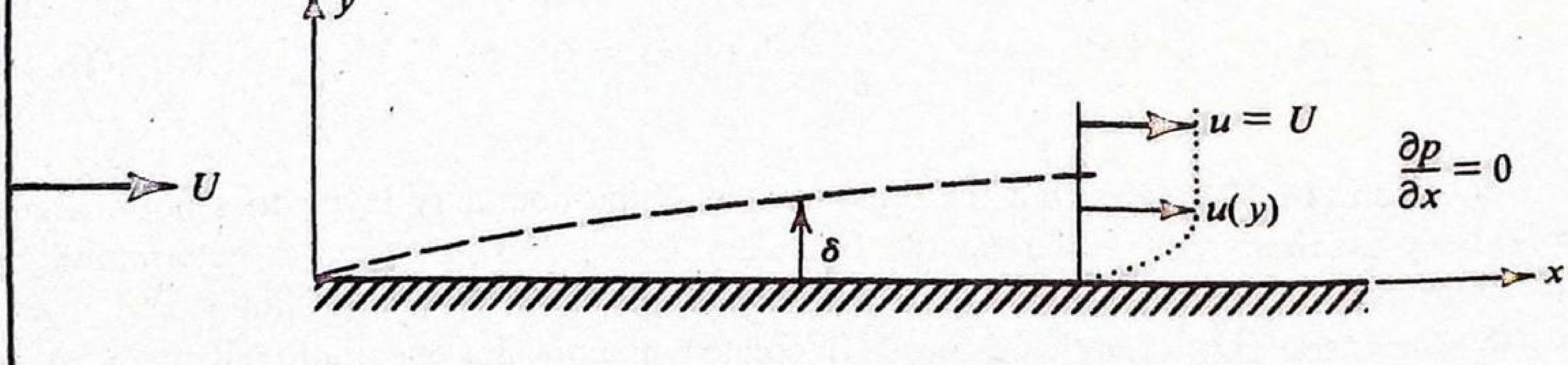

Figure E14.1

*For turbulent flow we add the turbulent acceleration $-\ [\partial(\overline{u'v'})/\partial y]$.

**See Ref. 14.1 for details of the mathematical analysis.

Example 14.1 *(Con't.)*

Solution:

The radius of curvature ρ of the distribution of velocity $u(y)$ is that used in the calculus:

$$\rho = \frac{[1 + (du/dy)^2]^{3/2}}{|d^2u/dy^2|} \quad \text{(i)}$$

The boundary conditions at the surface of the flat plate are

$$u = v = 0 \text{ at } y = 0 \quad \text{(ii)}$$

Substituting the above boundary conditions into the Prandtl boundary layer Eq. (14.11) yields

$$\overset{0}{u}\frac{\partial u}{\partial x} + \overset{0}{v}\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\overset{0}{\partial p}}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{(iii)}$$

resulting in

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0 \quad \text{(iv)}$$

Substituting the gradient of the shear stress of Eq. (iv) into the expression for the radius curvature of Eq. (i) gives

$$\rho = \infty \quad \text{(v)}$$

which means that very close to the surface of the plate, the velocity is linear and the shear stress is constant.

This completes the solution.

Example 14.2

Reduce the Prandtl boundary layer equations to a simpler form than that given by Eqs. (14.11)–(14.13) for (a) flow over a flat plate, (b) the case $p_{yx} = c_1$ (a constant), and (c) the case where $v \propto \nu$. (d) Solve the Prandtl boundary layer equations for the special case $v = \nu$ and where the pressure gradient $\partial p/\partial x$ is zero.

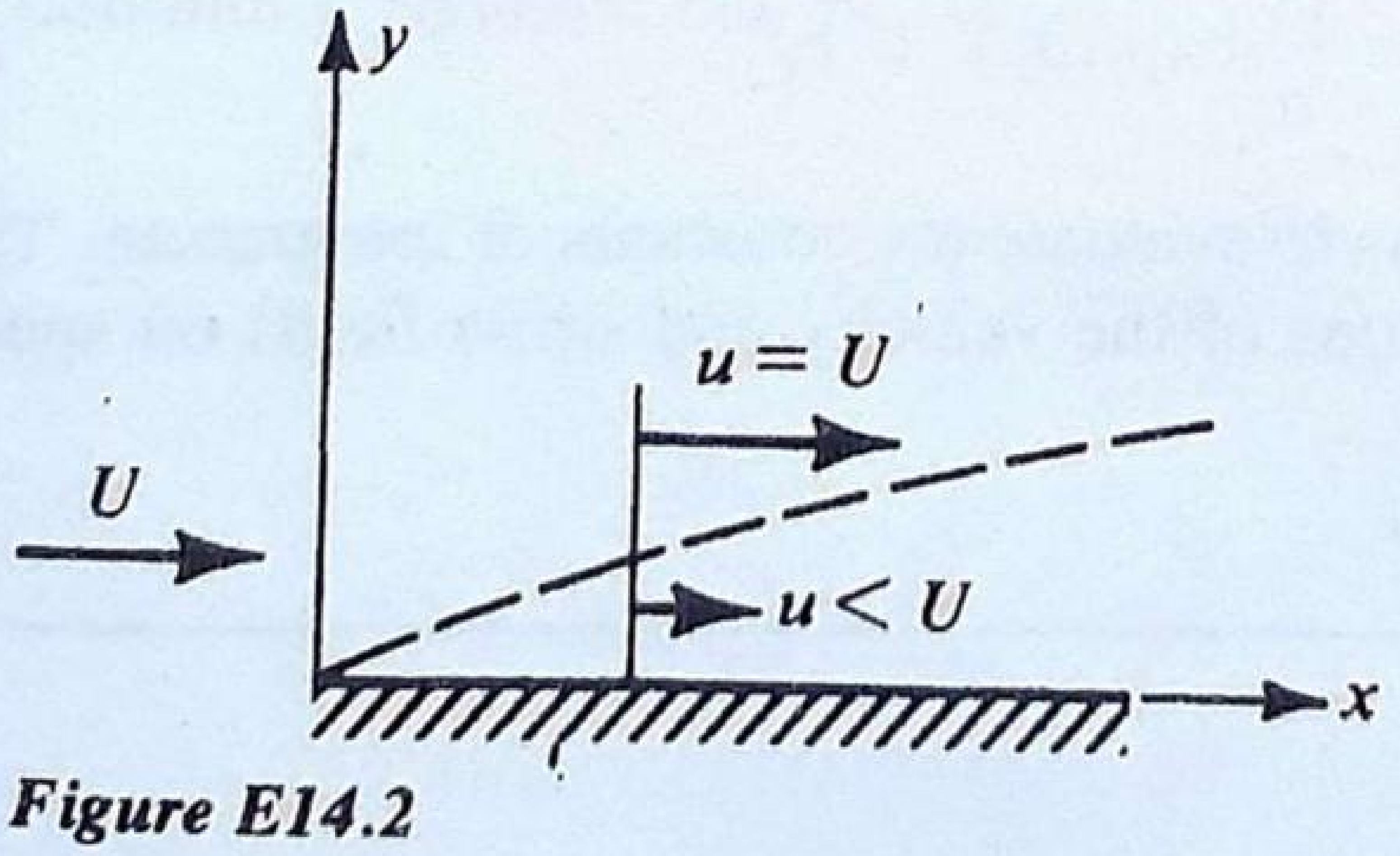

Figure E14.2

Example 14.2 *(Con't.)*

Solution:

(a) For flow past a flat plate, $\partial p/\partial x = 0$, Eq. (14.11) reduces to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(i)}$$

Equation (i) is the partial differential equation H. Blasius solved for his Ph.D. dissertation in Göttingen, Germany [14.2].

(b) For the constant shear stress case, Eq. (14.11) reduces to

$$u\frac{\partial u}{\partial x} + \frac{c_1}{\mu}v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(ii)}$$

which can be altered to yield

$$p + \frac{1}{2}\rho u^2 = -\frac{c_1}{\nu}\int v dx \qquad \text{(iii)}$$

Thus the total pressure can be determined if we know how the y-component of velocity v varies in the flow.

(c) For the case $v \propto \nu$, the Prandtl boundary layer equations reduce to

$$\frac{d^2u}{dy^2} - a\frac{du}{dy} = -\frac{1}{\mu}\frac{dp}{dx} \qquad \text{(iv)}$$

where a is a constant of proportionality. We note that the left-hand side is a function of y, and the right-hand side is a function of x, which signifies both the left- and right-hand side terms are constant.

(d) Setting dp/dx to zero, Eq. (iv) reduces to

$$\frac{d(du/dy)}{du/dy} = ady \qquad \text{(v)}$$

Integrating Eq. (v) and taking antilogs of both sides yields

$$\frac{du}{dy} = c_1 \exp(ay) \qquad \text{(vi)}$$

Integrating Eq. (vi) yields

$$u = \frac{c_1}{a}\exp(ay) + c_2 \qquad \text{(vii)}$$

We need two boundary conditions to evaluate the constants of integration. These would stem from known conditions of the velocity and stress fields on specific boundaries that define the flow.

This completes the solution.