Next lecture :- Lecture 8

$$u = u f'$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = \frac{V f''}{\delta_0}$$

$$\frac{\partial u}{\partial x} = -\frac{u \delta_0'}{\delta_0} \eta f'' , \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 \psi}{\partial y^3} = \frac{u f'''}{\delta_0^2}$$

where the symbols $\delta_0' f'$ are defined as

$$\delta_0' = \frac{d\delta_0}{dx} , \quad f' = df/d\eta$$

by substituting in (1) for the similarity variables into the transformed prandtl layer equation

$$\Rightarrow f''' + \frac{u}{V} \delta \delta_0' f f'' = 0$$

Since $\eta$ and $x$ are independent variables for similar profiles then the co-efficients $\frac{U}{V}\delta_0\delta_0'$ must be a constant.

$$\frac{U}{V}\frac{\delta_0 d\delta_0}{dx} = C$$

integrating;

$$\frac{U}{V}\frac{\delta_0^2}{2} = Cx + C_1$$

Choosing a co-ordinate origin such that $\delta_0 = 0$ at $x = 0 \Rightarrow C_1 = 0$ and thus the boundary layer thickness $\delta_0$ is found to be

$$\delta_0 = \sqrt{2C\frac{x\nu}{U}}$$

Since the constant $C$ can be any value, Blasius conveniently choose $C = 1/2$ such that the similarity variable $\eta$ was defined as

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

The prandtl boundary layer equation for uniform flow over a flat plate becomes now an ordinary non-linear differential equation

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \quad \text{--- (2)}$$

$$f' = f = 0 \text{ at } \eta = 0$$

$$f' \rightarrow 1 \text{ as } \eta \rightarrow \infty$$

To solve boundary value problem thus posed, Blasius assumed a series for the similarity function 'f' in power series form

$$f(\eta) = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \cdots + \frac{A_m}{m!}\eta^m \quad (3)$$

where $A_i$ are undetermined co-efficients to be determined from the boundary conditions,

$$f = f' = 0 \text{ at } \eta = 0 \quad , \quad f' \to 1 \text{ as } \eta \to \infty$$

Since there are only three independent boundary conditions, only three co-efficients of $A_i$ can be found, and all other co-efficients must be related to these three co-efficients. It is quite simple to show that two boundary condition

$$f = f' = 0 \Rightarrow A_0 = 0, \ A_1 = 0$$

$$f(0) = A_0 = 0 \quad , \quad f'(\eta) = A_1 + \frac{2A_2}{2!}\eta + \cdots$$

$$f'(0) = A_1 \Rightarrow A_1 = 0$$

Substituting the power series expression for the similarity function of eq (3) into prandtl equation (2). Blasius obtained after some simple algebra the series expression $f''' = ?$, $f'' = ?$, $f = ?$

$$2A_5 + 2A_4\eta + (A_2^2 + 2A_5)\frac{\eta^2}{2!} + (4A_2A_5 \cancel{A_3})\frac{\eta^3}{3!} + \cdots$$

$$A_5 = A_3 = A_6 = A_7 = A_9 = A_1 = \cdots \quad (1a)$$

$A_5 = -\frac{1}{2} A_2^2$, $A_8 = -\frac{11}{2} A_2$, $A_5 = \frac{11}{4} A_2^3$

$$f(\eta) = \sum_{m=0}^{\infty} \left(-\frac{1}{2}\right)^m \frac{A_2^{m+1} C_m}{(3m+2)!} \eta^{3m+2}$$

$C_0 = 1$, $C_1 = 1$, $C_2 = 11$, $C_3 = 375$, $C_4 = 27,827$

$C_5 = 3,817,137, \ldots$

$$f(\eta) = \frac{\eta^2}{2} - \frac{\eta^5}{2 \cdot 5!} + \frac{C_2 \eta^8}{4 \cdot 8!} - \frac{C_3 \eta^{11}}{8\, 11!} + \frac{C_4 \eta^{14}}{16 \cdot 14}$$

Here $A_2$ is the only unknown co-efficient in the power series operator and must be determined from the boundary condition

$f' \to 1$ as $\eta \to \infty$ i.e. $A_2 = 1$

above leads to result

$$f(\eta) = A_2^{1/3} \phi (A_2^{1/3} \eta)$$

| $\eta$ | $f(\eta)$ | $f'(\eta)$ | $f''(\eta)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.33206 |
| 0.2 | 0.00681 | 0.06641 | 0.33198 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 8.0 | 6.279 | 1.0 | 0.00243 |

## Potential Flow

The flow that are incompressible and irrotational. Further, the use of the velocity potential to describe the fluid flow is a popular technique called potential flow.

The continuity equation in differential form for an incompressible fluid is given by

$$\nabla . V = 0 \quad — ①$$

If we impose the condition of irrotational then

$$\nabla \times V = 0 \quad — ②$$

At solution of the differential equation is the velocity potential $\phi$ given by

$$V = \nabla \phi \quad — ③$$

which when distributed into eq ①

$$\nabla . \nabla \phi = \nabla^2 \phi = 0 \quad — ④$$

This is called laplacis equation. If we wish to express Laplaciu equation in cartesian co-ordinate eq ④ becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

using the cylindrical co-ordinates we express ④ as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The velocity potential $\phi$ for every possible irrotational motion of incompressible fluid flow must satisfy the Laplace eq ④.

Note:- Most ideal fluid motion are irrotational.

Ex: Show that $\phi = x^3t + 2y^2t - 3txz^2 - 2z^2t$ is a possible velocity potential for a three-dimensional incompressible potential fluid flow.

__Solution:__ The fluid flow is three-dimensional unsteady, incompressible and irrotational. The appropriate governing eq. of motion is Laplace's equation

$$\nabla^2\phi = 0$$

$$\phi = x^3t + 2y^2t - 3txz^2 - 2z^2t$$

$$\frac{\partial\phi}{\partial x} = 3x^2t - 3tz^2 \quad , \quad \frac{\partial^2\phi}{\partial x^2} = 6xt \quad , \quad \frac{\partial\phi}{\partial y} = 4yt$$

$$\frac{\partial^2\phi}{\partial y^2} = 4t \quad , \quad \frac{\partial\phi}{\partial z} = -6txz - 4zt$$

$$\frac{\partial^2\phi}{\partial z^2} = -6xt - 4t$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \cancel{6xt} + \cancel{4t} - \cancel{6xt} - \cancel{4t}$$

$$= 0$$

Thus Laplacis eq. is satisfied and the flow is irrotational.

Ex:- Given the velocity field $V = 6i + 8j - 10k$

a) Show that flow is irrotational

b) Find the velocity potential $\phi$.

Sol:- The flow is three-dimensional steady and incompressible

The appropriate eq. is Laplace equation

$$\nabla^2 \phi = 0$$

Define of velocity potential

$$V = \nabla \phi$$

a) To show that flow is indeed irrotational, we must show that vorticity $\xi$ vanishes; i-e

$$\xi = \nabla \times V$$

$$= \nabla \times (6i + 8j - 10k)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(6i + 8j + 10k)$$

$$= 0$$

b) $V = 6\hat{i} + 8\hat{j} - 10\hat{k} \Rightarrow u = 6, v = 8, w = -10$

$u = \frac{\partial \phi}{\partial x} = 6$, $v = \frac{\partial \phi}{\partial y} \Rightarrow \frac{\partial \phi}{\partial y} = 8$

$$w = \frac{\partial \phi}{\partial z} = -10$$

$$\phi = 6x + f_1(y) + f_2(z)$$

$$\phi = 8y + f_3(x) + f_4(z)$$

$$\phi = -10z + f_5(x) + f_6(y)$$

by comparing

$f_1(y) = f_6(y) = 8y \qquad f_3(x) = f_5(x) = 6x$

$f_2(z) = f_4(z) = -10z$

(4))

$\phi = 6x + 8y - 10z$ is the velocity potential