Blausius Solution For laminar Boundary Layer Flow over a Flate plate.

The classical Problem. Blausius considered was a two dimensional steady, incompressible flow over a flat plate at zero angle of incidence w.r.t the uniform coming stream of velocity U. The fluid extends to infinity a in all ~~domain~~ direction from the plate as shown from above figure.

Findings: Blausius wanted to determine

→ Velocity field within the boundary layer

→ Boundary layer thickness δ.

→ shear stress distribution on the plate.

→ drag force on the plate.

Since the plate is flat and laminar has negligible thickness and the stream velocity U is uniform, the pressure gradient $\frac{\partial P}{\partial x}$ must vanish. The resultant prandtl boundary layer equation is then

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad \text{—} \ 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{—} \ 2$$

To solve the problem, first sought to reduce the number of variables. The continuity eq (2) can be satisfied by the stream function ψ as $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Substituting the stream function velocity relation-ship into prandtl boundary layer.

$$\Rightarrow \frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}$$

which is a third-order, non-linear, partial differential equation with now a single unknown function ψ.

A large class of boundary layer problem is solved by transforming the partial differential equation into ODE. The PDE can be transformed into an ODE then the dependent variable must be a function of a single variable η i.e:

$$u = U F(\eta) \quad , \quad v = U G(\eta)$$

where $F(\eta)$ and $G(\eta)$ are dimensionless similarity functions and η is dimensionless similarity variable. Blausius defined a similarity variable η as $\eta = y/\delta_0$

where δ_0 is proportionality to the boundary layer thickness δ and is a function of x only.

If we set $\psi = 0$ on the plate which is equivalent to $\eta = 0$ then the stream function ψ is evaluated from equation of continuity as

$$\psi = \int_0^y u\,dy$$

Blausius expressed the above integral and in term of the similarity function

$$\psi = U \int_0^\eta F(\eta)\, \delta_0\, d\eta$$

Skip for understanding;

$$\eta = y/\delta_0 \qquad , \eta = 0 \text{ at } y = 0.$$

$$\frac{d\eta}{dy} = \frac{1}{\delta_0} \qquad \eta = 1 \text{ at } y = \delta_0$$

$$dy = \delta_0 \, d\eta$$

So $\psi = U \int_0^{\eta} F(\eta) \, \delta_0 \, d\eta$

$$\psi = U \delta_0 \int_0^{\eta} F(\eta) \, d\eta = U \delta_0 f(\eta)$$

where $f(\eta) = \frac{dF}{d\eta} = F'$