CS 441 Discrete Mathematics for CS
Lecture 19

Probabilities

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Probabilities

- **Experiment:**
  - a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(E)= |E| / |S|**
- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example:

- roll of two dices
- What is the probability that the outcome is 7.
- All possible outcomes (sample space S):
- (1,6) (2,6) …(6,1),…(6,6) total: 36

- Outcomes leading to 7 (event E)
- (1,6) (2,5) …(6,1) total: 6
- P(sum=7)= 6/36 = 1/6

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Winning combination (event E): 1
- Probability of winning:
  - P(E) =1/C(40,6)= 34! 6! / 40! = 1/3,838,380

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example (cont):

- Odd of winning a second prize in lottery: hit 5 of 6 numbers selected from 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Second prize (event E): C(6,5)*(40-6)= 6*34
- Probability of winning:
  - P(E) =6*34/C(40,6)= (6*34)/3,838,380

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example (cont):

- Odd of winning a **third prize** in lottery: hit 4 of 6 numbers selected from 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Third prize (event E): C(6,4)*C(40-6,2) = C(6,4)*C(34,2)
- Probability of winning:
  - P(E) = C(6,4)*C(34,2) /C(40,6)

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Another lottery:

- 6 numbers (ordered) selected out of 40 numbers (with repetitions)
- Total number of outcomes:
  - Permutations with repetitions: $= 40^6$
- Number of winning configuration: 1
  - $P(win) = 1/40^6$

And its modification:

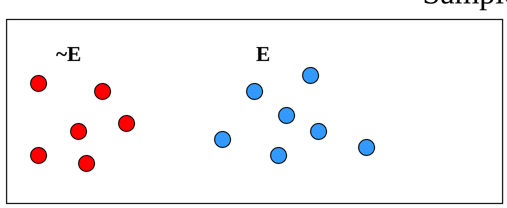
- If the winning combination is order independent:
  - E.g. (1,5,17,25,5,13) is equivalent to (5,17,5,1,25,13)
  - Number of winning permutations = number of permutations of 6 = 6!
  - $P(win) = 6! / 40^6$

Probabilities

Theorem: Let E be an event and ~E its complement with regard to S. Then:

- P(~E) = 1 – P(E)

Sample space



Proof.

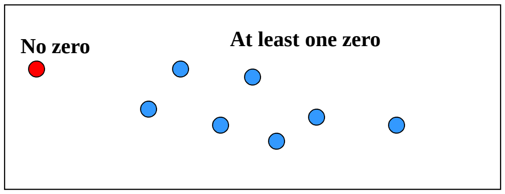
P(~E) = (|S|- |E|) /| S | = 1- |E| / |S|

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

All strings



- Event: seeing no-zero string   P(E) =?
- ~Event: seeing at least one zero in the string

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Probabilities

Example:

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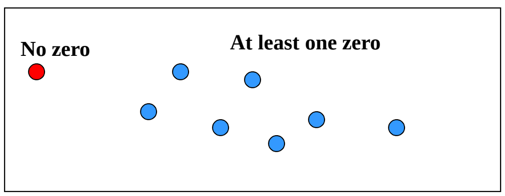
All strings



- Event: seeing no-zero string $P(E) = 1/2^{10}$
- ~Event: seeing at least one zero in the string
  $P(\sim E) = 1 - P(E) = 1 - 1/2^{10}$

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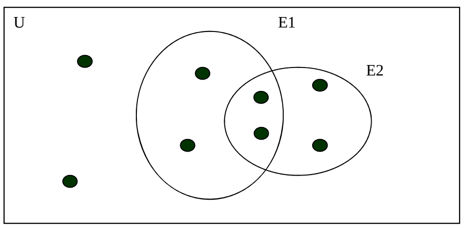
Probabilities

Theorem. Let E1 and E2 be two events in the sample space S. Then:

- $P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$

- This is an example of the inclusion-exclusion principle



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Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- P(E1)= 50/100
- P(E2)= 20/100
- $P(E1 \cap E2)$ = 10/100
- $P(E1 \cup E2)$ = (5+2-1)/10 = 6/10

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Probabilities

- Assumption applied so far:
  - **the probabilities of each outcome are equally likely.**
- However in many cases outcomes may not be equally likely.

Example: a biased coin or a biased dice.

- **Biased Coin:**
  - Probability of head 0.6,
  - probability of a tail 0.4.
- **Biased Dice:**
  - Probability of **6:** 0.4,
  - Probability of **1, 2, 3, 4, 5:** 0.12 each

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Probabilities

Three axioms of the probability theory:

(1) Probability of a discrete outcome is:

- $0 <= P(s) <= 1$

(2) Sum of probabilities of all (disjoint) outcomes is = 1

(3) For any two events E1 and E2 holds:

$$P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$$

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Probability distribution

Definition: A function **p: S → [0,1]** satisfying the three conditions is called **a probability distribution**

Example: a biased coin

- Probability of head 0.6, probability of a tail 0.4
- **Probability distribution:**
  - Head → 0.6
  - Tail → 0.4

The sum of the probabilities sums to 1

Note: **a uniform distribution** is a special distribution that assigns equal probabilities to each outcome.

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Probability of an Event

Definition: The probability of the event *E* is the sum of the probabilities of the outcomes in *E*.

$$P(E) = \sum_{s \in E} P(s)$$

- Note that now no assumption is being made about the distribution.

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Example

Probability of an event $$P(E) = \sum_{s \in E} P(s)$$

Example: Suppose that a die is biased so that 3 appears twice as often as each other number, but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

Solution: We want the probability of the event $E = \{1,3,5\}$. Probabilities of outcomes:

- $p(3) = 2/7$ and
  $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$.

Then, $p(E) = p(1) + p(3) + p(5) =$
$1/7 + 2/7 + 1/7 = 4/7$.

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Probabilities of Complements and Unions

- Complements still hold. Since each outcome is in either E or $\overline{E}$ but not both,

$$p(\overline{E}) = 1 - p(E)$$

- Unions: $\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E})$.

also still holds under the new definition.

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

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Conditional probability

Definition: Let E and F be two events such that $P(F) > 0$. The **conditional probability** of E given F

- $P(E|F) = P(E \cap F) / P(F)$

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Conditional probability

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Example:

- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- **Probability of having two boys P(BB) = ¼**
- **Probability of having one boy P(one boy) = ¾**
- **P(BB|given a boy) = ¼ / ¾ = 1/3**

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BB BG GB GG

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Conditional probability

Corrolary: Let E and F be two events such that P(F) > 0. Then:

- $P(E \cap F) = P(E| F) * P(F)$

Proof:

- From the definition of the conditional probability:

$P(E|F) = P(E \cap F) / P(F)$

→

$P(E \cap F) = P(E|F) P(F)$

- **This result is also referred to as the product rule.**

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Conditional probability

Corrolary: Let E and F be two events such that $P(F) > 0$. Then:

- $P(E \cap F) = P(E|F) P(F)$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a high fever given the flu: 0.9

What is the probability of getting a flu with fever?

$P(flu \cap fever) = P(fever \mid flu)P(flu) = 0.9*0.2 = 0.18$

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- When is this useful?

Sometimes conditional probabilities are easier to estimate.

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