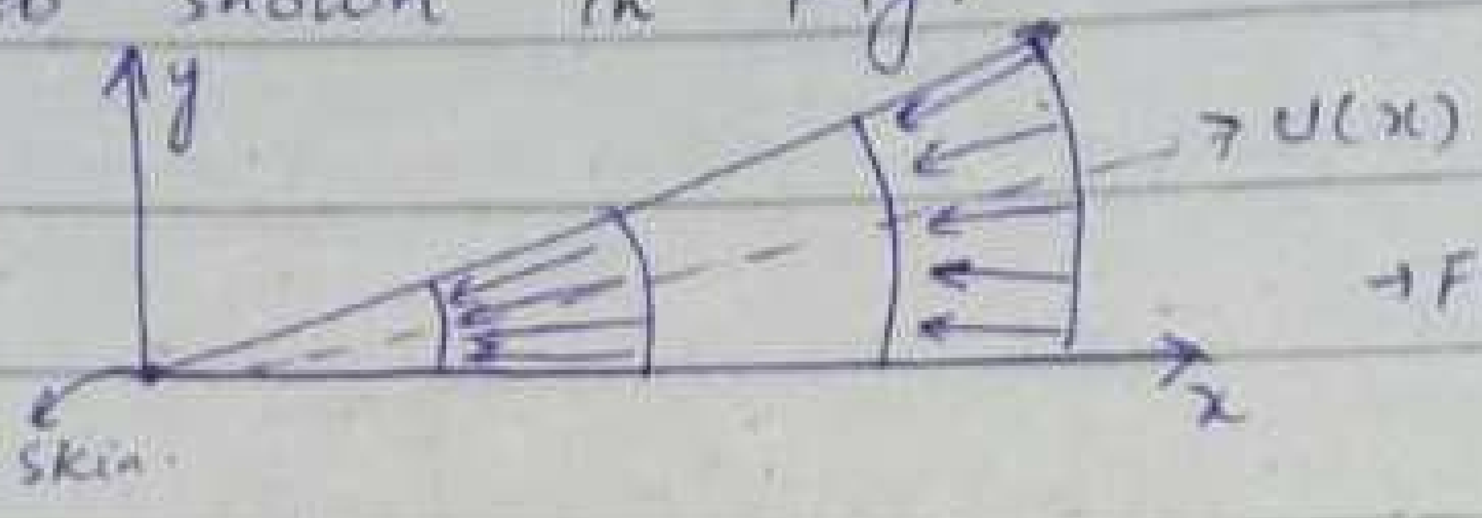
31/3/20 Lecture 5

Flow in a convergent channel

Consider two dimensional steady state flow of viscous incompressible fluid in a convergent channel as shown in Fig.



→ Flow in convergent channel.

The system of boundary layer equations established the fluid flow model in convergent channel is given as below

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad -(1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad -(2)$$

B.Cs: $u=0$, $v=0$ at $y=0$
$u \to U(x)$, as $y \to \infty$. } (3)

the case of potential flow given by

$$U(x) = -\frac{U_1}{x} \quad -(4)$$

as related to flow past a wedge, and also leads to similar solutions. with $U_1 > 0$ it represents two dimensional motion in convergent channel with flat walls (sink). The Volume flow rate for a full opening angle 2α and for a stratum of unit height $Q = 2\pi U_1$. There we introduce the Similarity transformation.

1) Group of transformation for convergent channel:-

$$\left.\begin{aligned} \psi(x,y) &= -\sqrt{\nu U_1}\, f(\eta) \\ \eta &= \frac{y}{x} = \sqrt{\frac{Q}{2\pi\nu}} \end{aligned}\right\} \quad (5)$$

Now, by using this group of transformation. we will find the velocity component u and v by using

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} \quad \text{—— (6)}$$

$$u = -\frac{\partial}{\partial y}\left(\sqrt{\nu U_1}\, f(\eta)\right)$$

$$= -\sqrt{\nu U_1}\,\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial y}, \quad \frac{\partial\eta}{\partial y} = \frac{1}{x}\sqrt{\frac{Q}{2\pi\nu}}$$

$$u = -\frac{\sqrt{\nu U_1}}{x}\cdot f'\sqrt{\frac{Q}{2\pi\nu}} \quad \because Q = 2\pi U_1$$

$$U = -\frac{\sqrt{U_1}}{x} f'\sqrt{\frac{2\pi U_1}{2\pi}} = -\frac{U_1}{x} f'$$

$$U_1 = -Ux\cancel{f}$$

$$\Rightarrow u = \frac{Ux}{x} f' \Rightarrow u = Uf' \quad \text{—— (7)}$$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\left(-\sqrt{\nu U_1}\, f(\eta)\right)$$

$$v = \frac{\partial}{\partial x}\left(\sqrt{\nu u_1}\, f(\eta)\right)$$

$$= \sqrt{\nu u_1}\,\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial x} = -\frac{y}{x^2}\sqrt{\frac{Q}{2\pi\nu}} = -\frac{1}{x}\cdot\eta$$

$$v = \sqrt{\nu u_1}\, f' - \frac{\eta}{x} \quad -(8)$$

Now $u = \bar{u} f'$ $\Rightarrow$ $U = \frac{-u_1}{x} f'$

$$\frac{\partial u}{\partial x} = u\frac{\partial f'}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{u_1}{x^2} f'$$

$$\frac{\partial u}{\partial x} = +u\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{u_1}{x^2}f' = -u\cdot\frac{\eta}{x}f'' + \frac{u_1}{x^2}f'$$

$$\frac{\partial u}{\partial x} = -\frac{\eta u f''}{x} + \frac{x u}{x x} f' \qquad U = -\frac{U_1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{u}{x}(-f' - \eta f'') \quad -(9)$$

$U = uf'$

$$u\frac{\partial u}{\partial x} = \frac{U^2}{x}(-f'^2 - \eta f' f'') \quad -(10)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(uf') = u\frac{\partial f'}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = uf''\cdot\frac{1}{x}\sqrt{\frac{Q}{2\pi\nu}}$$

$$\frac{\partial u}{\partial y} = \frac{u}{x}f''\sqrt{\frac{\cancel{2\pi} u_1}{\cancel{2\pi}\nu}} = \frac{u}{x}\sqrt{\frac{u_1}{\nu}}\,f''$$

$$\nu \frac{\partial u}{\partial y} = \left(\frac{u}{x} \sqrt{\frac{u_1}{\nu}} f'' \right) \left(-\sqrt{\nu u_1} f' \cdot \frac{\eta}{x} \right)$$

$$= -\frac{u}{x^2} \cdot u_1 \eta f' f''$$

$$= -\frac{u}{x^2} (-ux) \eta f' f''$$

$$= \frac{u^2}{x} \eta f' f'' \quad — (11)$$

Now, $\frac{\partial u}{\partial y} = \frac{u}{x} \sqrt{\frac{u_1}{\nu}} f''$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u}{x} \sqrt{\frac{u_1}{\nu}} \frac{\partial f''}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u}{x} \sqrt{\frac{u_1}{\nu}} f''' \cdot \frac{1}{x} \sqrt{\frac{\cancel{x} u_1}{\cancel{x} \nu}}$$

$$u(x) = -\frac{u_1}{x}$$

$$\frac{du}{dx} = \frac{u_1}{x^2} = -\frac{u}{x}$$

$$u\frac{du}{dx} = -\frac{u^2}{x}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{u}{x^2} \cdot \frac{ux}{\nu} f''' = -\frac{u^2}{\nu x} f''' \quad — (12)$$

Now by using (10) (12) into (2)

$$\frac{u^2}{x} (-f'^2 - \eta f' f'') + \frac{u^2}{x} \eta f' f'' = -\frac{u^2}{x} + \nu \left(-\frac{u^2}{x} \frac{1}{\nu} \right) f'''$$

$$\Rightarrow -\frac{u^2}{x} f'^2 - \cancel{\frac{u^2}{x} \eta f' f''} + \cancel{\frac{u^2}{x} \eta f' f''} = -\frac{u^2}{x} - \frac{u^2}{x} f'''$$

$$-\frac{\cancel{u^2}}{\cancel{x}} f'^2 = \frac{\cancel{u^2}}{\cancel{x}} (-1 - f''') \Rightarrow -f'^2 = -1 - f'''$$

$$f''' - f'^2 + 1 = 0$$

$$f''' \mp f'^2 + 1 \pm 0 \quad — (13)$$

Subject to the transformed boundary conditions

$f = f' = 0$ at $\eta = 0$, $f' \to 1$ as $\eta \to \infty$

So, $f''' + f'^2 + 1 = 0$, with B.Cs.

from the numerical solutions of the above give ordinary differential equation, we can find velocity at the fluid (f') within convergent channel and the skin friction quantity (f'') with in the convergent channel.

Multiplying eq (13) by f''

$f''f''' + f''f'^2 + f'' = 0$

Assignment :-

$f''f''' - f''f'^2 + f'' = 0$ — (A)

$\frac{f''^2}{2} - \int f'^2 f'' d\eta + \int f'' d\eta = a'$

$\left(\frac{f''^2}{2} - \frac{f'^3}{3} + f' = a'\right) \times 2$

$f''^2 - \frac{2f'^3}{3} + 2f' = 2a'$

$f''^2 - \frac{2}{3}(f'^3 - 3f') = 2a'$

$f''^2 - \frac{2}{3}[f'^3 - 2f' - f' - 2 + 2] = 2a'$

$f''^2 - \frac{2}{3}[f'^3 - 2f' + 2f' - 2f' - f' + 2 - 2] = 2a'$

$\cancel{f''^2 - \frac{2}{3}(f'^3 + f' + 2f'^2 +}$

$$f''^2 - \frac{2}{3}\left[f'^3 - 4f' + f' + 2\right] + \frac{4}{3} = 2a' \qquad a = 2a' - \frac{4}{3}$$

$$f''^2 - \frac{2}{3}\left[f'^3 + f' + 2f'^2 + 2 - 2f'^2 - 4f'\right] = a$$

$$f''^2 - \frac{2}{3}(f'-1)^2(f'+2) = a$$

$$f''^2 - \frac{2}{3}(1-1)^2(1+2) = 0 \qquad f' = 1$$

$$f''^2 = 0 \Rightarrow f'' = 0$$

$f'' \to 0$ when $f' \to 1$ for $\eta \to \infty$

$$f''^2 = \frac{2}{3}(f'-1)^2(f'+2)$$

$$\frac{d^2f}{d\eta^2} = f''(\eta) = \left[\frac{2}{3}(f'-1)^2(f'+2)\right]^{1/2}$$

$$\frac{df'}{d\eta} = \sqrt{\frac{2}{3}(f'-1)^2(f'+2)}$$

$$\sqrt{\frac{3}{2}}\frac{df'}{\sqrt{(f'-1)^2(f'+2)}} = d\eta$$

$$\eta = \sqrt{2}\left\{\tanh^{-1}\left(\frac{\sqrt{2+f'}}{\sqrt{3}}\right) - \tanh^{-1}\left(\sqrt{\frac{2}{3}}\right)\right\} \quad (B)$$

$$\eta = \sqrt{2}\left\{\tanh^{-1}\frac{\sqrt{2+\frac{u}{U}}}{\sqrt{3}} - 1.146\right\} \qquad \because f' = \frac{u}{U}$$

$$\tanh^{-1}\left(\frac{2}{3}\right)^{1/2} = 1.1 \quad 1.146.$$

$$\sqrt{3}\left(\frac{\eta}{\sqrt{2}} + \frac{1.146\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\right) = \tanh^{-1}\sqrt{2+\frac{u}{U}}$$

$$\left(\sqrt{2+\frac{u}{U}}\right)^2 = \left(\sqrt{3}\tanh\left(\frac{\eta}{\sqrt{2}} + 1.146\right)\right)^2$$

$$2 + \frac{u}{U} = 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + 1.146\right)$$

$$f' = \frac{u}{U} = 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + 1.146\right) - 2$$

Now introducing polar angle $\theta = y/x$ since we know previously,

$$\eta = \frac{y}{x}\sqrt{\frac{Q}{2\pi \nu}} \qquad , \theta = y/x$$

as well as else $Q = 2\pi r U$, $r =$ radial dist. from sink

$$\eta = \theta\sqrt{\frac{2\pi r U_r}{2\pi \nu}} = \theta\sqrt{\frac{r U_r}{\nu}}$$

$$\eta = \theta\sqrt{\frac{2\pi r U}{2\pi \nu}} = \theta\sqrt{\frac{r U}{\nu}}$$