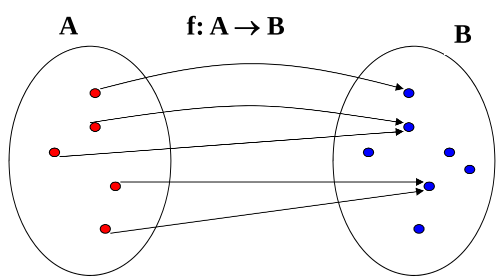
CS 441 Discrete Mathematics for CS
Lecture 9

Functions II

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \rightarrow B$** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Functions

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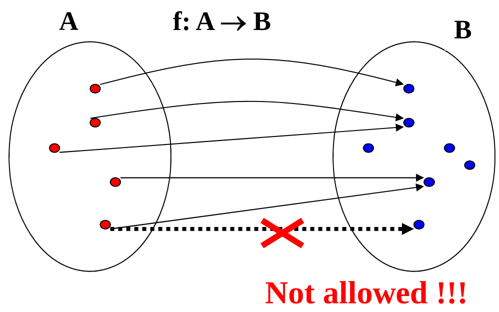
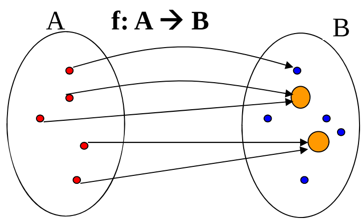


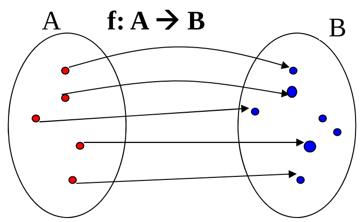
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternative: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.



Not injective function

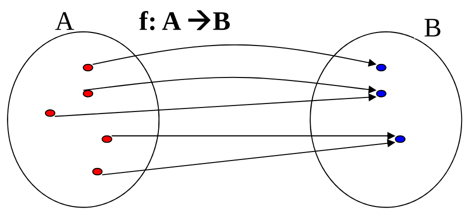


Injective function

Surjective function

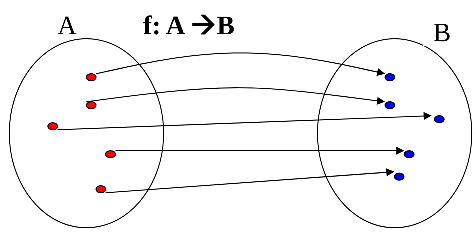
Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one (injection) and onto (surjection)**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **?**

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Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

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Bijective functions

Example 2:

- Define g : W → W (whole numbers), where $g(n) = \lfloor n/2 \rfloor$ (floor function).
  - $0 \to \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
  - $1 \to \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
  - $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...
- Is g a bijection?

Bijective functions

Example 2:

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  - $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...
- Is g a bijection?
  - **No.** g is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.

Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

→ **A is finite** and **f is one-to-one (injective)**

- Is f an **onto function (surjection)**?

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Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?

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Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Bijective functions

Theorem. Let f be a function from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- **Example:**
  - f : Z → Z, where f(z) = 2 * z.
  - f is one-to-one but not onto.
    - 1 → **2**
    - 2 → **4**
    - 3 → **6**
  - 3 has no pre-image.

Functions on real numbers

Definition: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- (f1 + f2)(x) = f1(x) + f2(x)
- (f1 * f2)(x) = f1(x) * f2(x).

Examples:

- **Assume**
  - f1(x) = x - 1
  - f2(x) = x^3 + 1

  then
  - (f1 + f2)(x) = x^3 + x
  - (f1 * f2)(x) = x^4 - x^3 + x - 1.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Example:

- Let g : **R** → **R**, where g(x) = 2x - 1. Is it increasing ?

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Example:

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?
- **Proof .**

  For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

  Thus g is strictly increasing.

Increasing and decreasing functions

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Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

Increasing and decreasing functions

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Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A: A \rightarrow A$ where $i_A(x) = x$.

Example:

- Let A = {1,2,3}

Then:

- $i_A(1) = ?$

Identity function

Definition: Let A be a set. The **identity function** on A is the function i_A: $A \rightarrow A$ where $i_A(x) = x$.

Example:

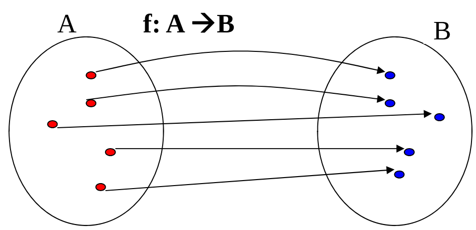
- Let A = {1,2,3}

Then:

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

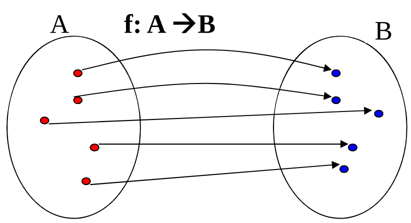
Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.

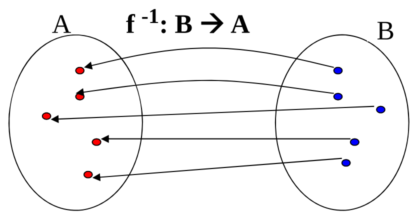


Inverse functions

Definition: Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible.**



f is bijective



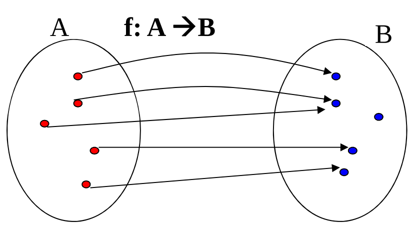
Inverse of f

Inverse functions

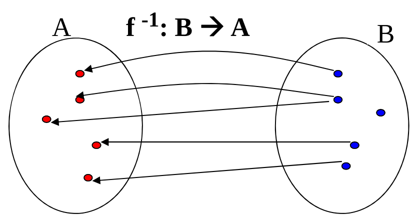
Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

?



f



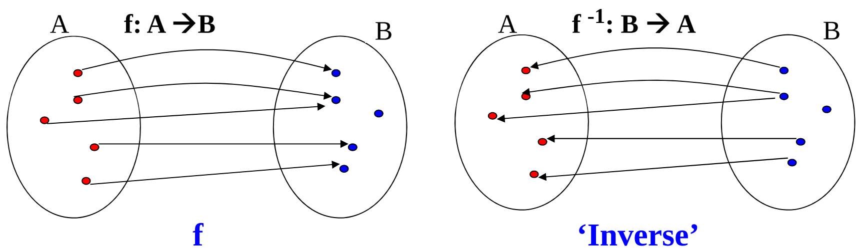
'Inverse'

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.



Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

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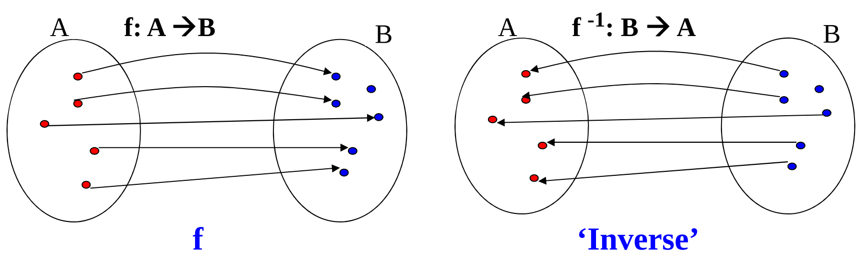
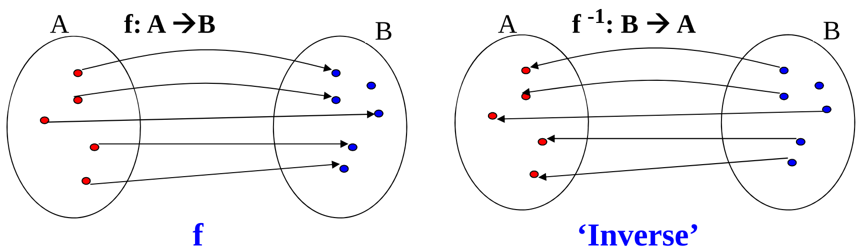


Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



Inverse functions

Example 1:

- Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\quad\quad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\quad\quad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\quad\quad$ $i_A^{-1}(3) = 3$

- Therefore, the inverse function of i_A is i_A.

Inverse functions

Example 2:

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function g^{-1} ?

Inverse functions

Example 2:

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define **g^{-1}(y) = x= (y+1)/2**

Test the correctness of inverse:

- **g(3)** = ..

Inverse functions

Example 2:

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function g^{-1} ?

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=> (y+1)/2 = x

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- **g(3)** = 2*3 - 1 = **5**
- $\mathbf{g^{-1}}$ **(5)** =

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Inverse functions

Example 2:

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Approach to determine the inverse:

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- **g(3)** = 2*3 - 1 = **5**
- $\mathbf{g^{-1}}$ **(5)** = (5+1)/2 = **3**
- g(**10**) =

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Inverse functions

Example 2:

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- What is the inverse function g^{-1} ?

Approach to determine the inverse:

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define **g^{-1}(y) = x= (y+1)/2**

Test the correctness of inverse:

- **g(3)** = 2*3 - 1 = **5**
- **g^{-1} (5)** = (5+1)/2 = **3**
- g(**10**) = 2*10 - 1 = **19**
- g^{-1} (19) =

Inverse functions

Example 2:

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define **g^{-1}(y) = x= (y+1)/2**

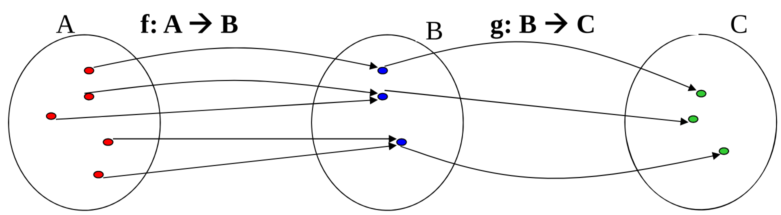
Test the correctness of inverse:

- **g(3)** = 2*3 - 1 = **5**
- **g^{-1} (5)** = (5+1)/2 = **3**
- g(**10**) = 2*10 - 1 = **19**
- g^{-1} (**19**) = (19+1)/2 = **10**.

Composition of functions

Definition: Let f be a function from set A to set B and let g be a function from set B to set C. The **composition of the functions g and f**, denoted by $g \circ f$ is defined by

- **$(g \circ f)(a) = g(f(a))$.**



Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \rightarrow A$,	$f: A \rightarrow B$
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

$f \circ g : A \rightarrow B$:

- $1 \rightarrow$

Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \rightarrow A,$	$f: A \rightarrow B$
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

$f \circ g : A \rightarrow B$:

- $1 \rightarrow d$
- $2 \rightarrow$

Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \rightarrow A,$	$f: A \rightarrow B$
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

$f \circ g : A \rightarrow B$:

- $1 \rightarrow d$
- $2 \rightarrow b$
- $3 \rightarrow$

Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \to A$,	$f: A \to B$
$1 \to 3$	$1 \to b$
$2 \to 1$	$2 \to a$
$3 \to 2$	$3 \to d$

$f \circ g : A \to B$:

- $1 \to d$
- $2 \to b$
- $3 \to a$

Composition of functions

Example 2:

- Let f and g be two functions from Z to Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \to Z$
- $(f \circ g)(x) = f(g(x))$
  $= f(x^2)$
  $= 2(x^2)$
- $g \circ f : Z \to Z$
- $(g \circ f)(x) =$?

Composition of functions

Example 2:

- Let f and g be two functions from Z to Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$
- $(f \circ g)(x) = f(g(x))$

  $= f(x^2)$

  $= 2(x^2)$
- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) = g(f(x))$

  $= g(2x)$

  $= (2x)^2$

  $= 4x^2$

Note that the order of the function composition matters

Composition of functions

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$

  $= f((x+1)/2)$

  $= 2((x+1)/2) - 1$

  $= (x+1) - 1$

  $= x$

Composition of functions

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$
  $= f((x+1)/2)$
  $= 2((x+1)/2) - 1$
  $= (x+1) - 1$
  $= x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
  $= f^{-1}(2x - 1)$
  $= (2x)/2$
  $= x$

Some functions

Definitions:

- The **floor function** assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)$ such that $1! = 1$