CS 441 Discrete Mathematics for CS
Lecture 8

Sets and set operations: cont.
Functions.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**
    V = { a, e, i, o, u }
  - **First seven prime numbers**.
    X = { 2, 3, 5, 7, 11, 13, 17 }

Sets - review

- A subset of B:
  - A is a subset of B if all elements in A are also in B.
- Proper subset:
  - A is a proper subset of B, if A is a subset of B and A ≠ B
- A power set:
  - The power set of A is a set of all subsets of A

Sets - review

- Cardinality of a set A:
  - The number of elements of in the set
- An n-tuple
  - An ordered collection of n elements
- Cartesian product of A and B
  - A set of all pairs such that the first element is in A and the second in B

Set operations

Set union:

- A = {1,2,3,6}   B = { 2,4,6,9}
- $A \cup B$ = { 1,2,3,4,6,9 }

Set intersection:

- A = {1,2,3,6}   B = { 2, 4, 6, 9}
- $A \cap B$ = { 2, 6 }

Set difference:

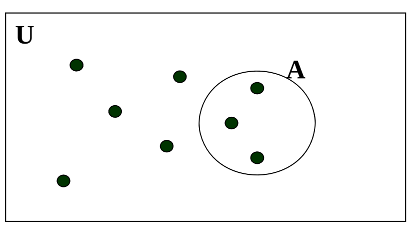
- A = {1,2,3,6}   B = { 2, 4, 6, 9}
- A - B = { 1, 3}
- B – A = {4, 9}

Complement of a set

Definition: Let U be the **universal set**: the set of all objects under the consideration.

Definition: The **complement of the set A**, denoted by $\bar{A}$, is the complement of A with respect to U.

- Alternate: $\bar{A} = \{ x \mid x \notin A \}$



Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A}$ = {2,4,6,8}

Set identities

Set Identities (analogous to logical equivalences)

- **Identity**
  - $A \cup \varnothing = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \varnothing = \varnothing$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \emptyset$

Set identities

- **Set identities can be proved using membership tables.**
- List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.
- Prove: $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Generalized unions and itersections

Definition: The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup ... \cup A_n\}$$

Example:

- Let A_i= {1,2,...,i} i =1,2,...,n
- $$\bigcup_{i=1}^{n} A_i = \{1, 2, ..., n\}$$

Generalized unions and intersections

Definition: The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap ... \cap A_n\}$$

Example:

- Let A_i = {1,2,...,i} i =1,2,...,n

$$\bigcap_{i=1}^{n} A_i = \{1\}$$

Computer representation of sets

- **How to represent sets in the computer?**
- **One solution: Data structures like a list**
- **A better solution:**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}
  - Computer representation: B = 10001

Computer representation of sets

Example:

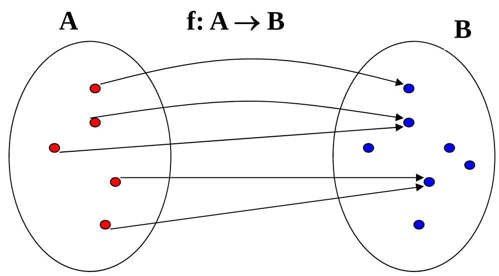
- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\bar{A} = 10110$

Functions

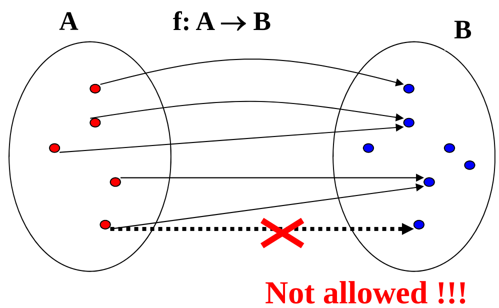
Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted $f : A \rightarrow B$, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \to B$** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



M. Hauskrecht

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
    - $1 \to c$
    - $2 \to a$
    - $3 \to c$
- Is f a function ?
- **Yes.** since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

M. Hauskrecht

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
  - $1 \rightarrow c$
  - $1 \rightarrow b$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is g a function?
- **No.** g(1) = is assigned both c and b.

M. Hauskrecht

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: $A \rightarrow B$ as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- $0 \rightarrow 0$ $\quad$ $3 \rightarrow 0$
- $1 \rightarrow 1$ $\quad$ $4 \rightarrow 1$
- $2 \rightarrow 2$ $\quad$ …

M. Hauskrecht

Important sets

Definitions: Let f be a function from A to B.

- We say that A is the **domain** of f and B is the **codomain** of f.
- If f(a) = b, **b is the image of a** and **a is a pre-image of b**.
- **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

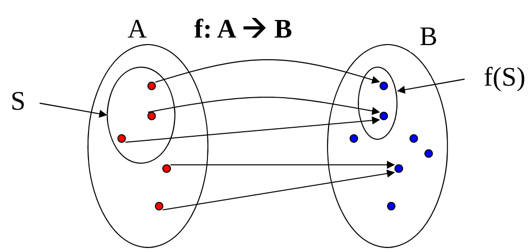
Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: $1 \rightarrow c$, $2 \rightarrow a$, $3 \rightarrow c$
- What is the image of 1?
- $1 \rightarrow c$ c is the image of 1
- What is the pre-image of a?
- $2 \rightarrow a$ 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

M. Hauskrecht

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



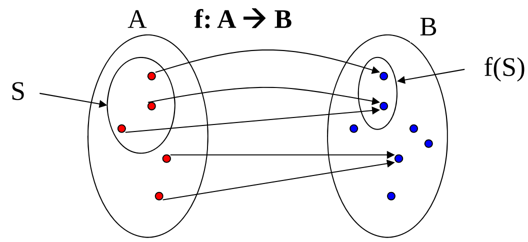
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: $1 \rightarrow c$, $2 \rightarrow a$, $3 \rightarrow c$
- Let S = {1,3} then image f(S) = **?**

M. Hauskrecht

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



Example:

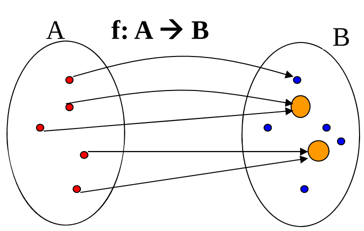
- Let A = {1,2,3} and B = {a,b,c} and f: $1 \rightarrow c$, $2 \rightarrow a$, $3 \rightarrow c$
- Let S = {1,3} then image f(S) = {c}.

M. Hauskrecht

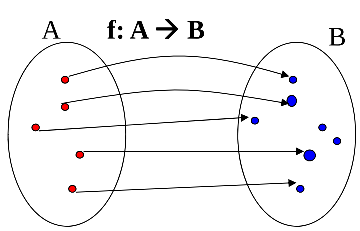
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternate: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.



Not injective



Injective function

M. Hauskrecht

Injective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and $1 \neq 3$.

Example 2: Let $g : Z \rightarrow Z$, where g(x) = 2x - 1.

- Is g is one-to-one (why?)
- **Yes.**
- Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b

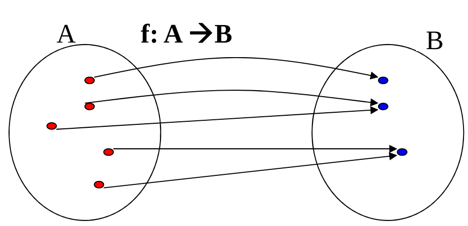
``   => a = b.

M. Hauskrecht

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that f(a) = b.

Alternative: all co-domain elements are covered



M. Hauskrecht

Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$

- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

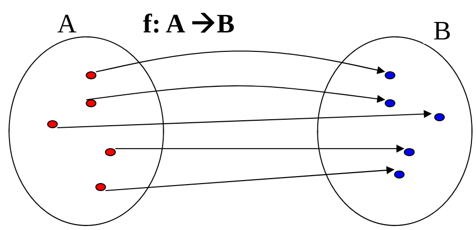
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \to B$ as h(x) = x mod 3.

- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - 1 $\rightarrow$ c
    - 2 $\rightarrow$ a
    - 3 $\rightarrow$ b
- Is f is a bijection? **Yes.** It is both one-to-one and onto.

- **Note:** Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.
- This is not true for A an infinite set. Define f : Z $\rightarrow$ Z, where f(z) = 2 * z. f is one-to-one but not onto (3 has no pre-image).

M. Hauskrecht

Bijective functions

Example 2:

- Define g : W $\rightarrow$ W (whole numbers), where g(n) = **[**n/2**]** (floor function).
  - 0 $\rightarrow$ **[**0/2**]** = **[**0**] =** 0
  - 1 $\rightarrow$ **[**1/2**]** = **[**1/2**]** = 0
  - 2 $\rightarrow$ **[**2/2**]** = **[**1**]** = 1
  - 3 $\rightarrow$ **[**3/2**]** = **[**3/2**]** = 1
- ...
- Is g a bijection?
  - **No.** g is onto but not 1-1 (g(0) = g(1) = 0 however 0 $\neq$ 1.

M. Hauskrecht