CS 441 Discrete Mathematics for CS
Lecture 7

Sets and set operations

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Basic discrete structures

- **Discrete math =**
  - study of the discrete structures used to represent discrete objects
- Many discrete structures are built using **sets**
  - **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    V = { a, e, i, o, u }
  - **First seven prime numbers**.

    X = { 2, 3, 5, 7, 11, 13, 17 }

Representing sets

Representing a set by:

1) Listing (enumerating) the members of the set.

2) Definition by property, using the set builder notation

{x| x has property P}.

Example:

- Even integers between 50 and 63**.**

  1) E = {50, 52, 54, 56, 58, 60, 62}

  2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**

Important sets in discrete math

- **Natural numbers:**
  - **N** = {0,1,2,3, …}
- **Integers**
  - **Z** = {…, -2,-1,0,1,2, …}
- **Positive integers**
  - $\mathbf{Z}^+$ = {1,2, 3.…}
- **Rational numbers**
  - **Q** = {p/q | p ∈ Z, q ∈ Z, q ≠ 0}
- **Real numbers**
  - **R**

Russell's paradox

Cantor's naive definition of sets leads to Russell's paradox:

- **Let S = { x | x ∉ x },**
  is a set of sets that are not members of themselves.
- **Question: Where does the set S belong to?**
  - Is S ∈ S or S ∉ S?
- **Cases**
  - **S ∈ S ?:** S does not satisfy the condition so it must hold that **S ∉ S (or S ∈ S does not hold)**
  - **S ∉ S ?:** S is included in the set S and hence S ∉ S does not hold
- **A paradox: we cannot decide if S belongs to S or not**
- **Russell's answer: theory of types – used for sets of sets**

Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- **{1,2,3} = {3,1,2} = {1,2,1,3,2}**

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

Example: Are {1,2,3,4} and {1,2,2,4} equal?

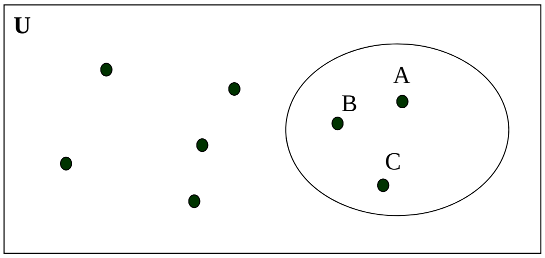
No!

Special sets

- **Special sets:**
  - **The universal set** is denoted by **U:** the set of all objects under the consideration.
  - **The empty set** is denoted as ∅ **or { }.**

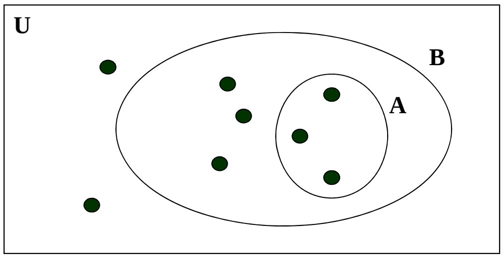
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use $A \subseteq B$ to indicate **A is a subset of B**.



- Alternate way to define A is a subset of B:

  $\forall x\ (x \in A) \rightarrow (x \in B)$

Empty set/Subset properties

Theorem $\emptyset \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \emptyset$ is **always False**
- Then the implication is **always True**.

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

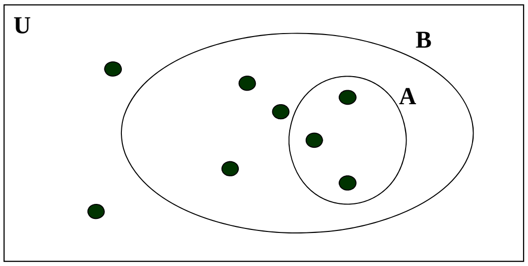
- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper subset

Definition: A **set A** is said to be a **proper subset** of B if and only if **$A \subseteq B$** and **$A \neq B$**. We denote that A is a proper subset of B with the notation $A \subset B$.



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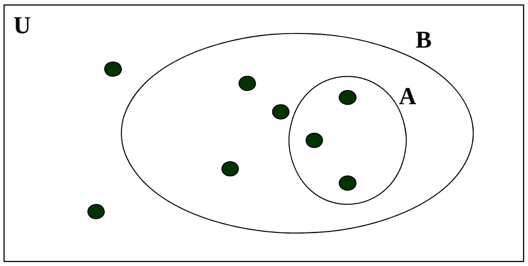

Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |.**

Examples:

- **V={1 2 3 4 5}**
  | V | = 5
- **A={1,2,3,4, …, 20}**
  |A| =20
- **| ∅ | = 0**

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }
- The set of reals is an infinite set.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set ∅
- What is the power set of ∅ ? P(∅) = { ∅ }
- What is the cardinality of P(∅) ? | P(∅) | = 1.

- Assume set {1}
- P({1}) = { ∅, {1} }
- |P({1})| = 2

Power set

- P({1}) = { ∅, {1} }
- |P({1})| = 2

- Assume {1,2}
- P({1,2}) = { ∅, {1}, {2}, {1,2} }
- |P({1,2})| =4

- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

- **If S is a set with |S| = n then | P(S) | = ?**

Power set

- P({1}) = { ∅, {1} }
- |P({1})| = 2

- Assume {1,2}
- P({1,2}) = { ∅, {1}, {2}, {1,2} }
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- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

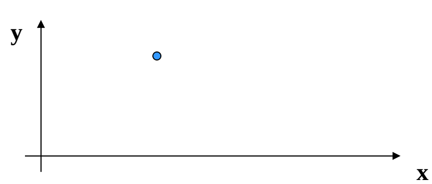
- **If S is a set with |S| = n then | P(S) | = 2^n**

N-tuple

- Sets are used to represent unordered collections.
- **Ordered-n tuples** are used to represent an ordered collection.

Definition: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, N ≥ 2.

Example:



- Coordinates of a point in the 2-D plane (12, 16)

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where s ∈ S and t ∈ T. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T\}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: S x T ≠ T x S **!!!!**

Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

Example:

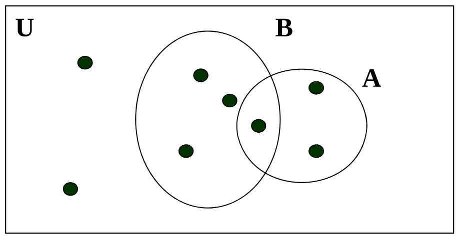
- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Set operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.

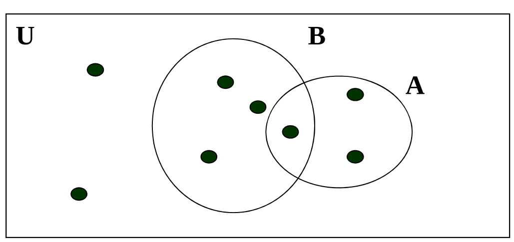


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{2,4,6,9\}$
- $A \cup B = \{1,2,3,4,6,9\}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



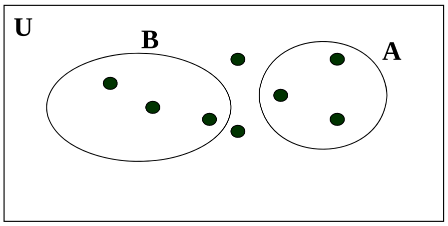
Example:

- $A = \{1,2,3,6\}$ $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2, 6\}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.



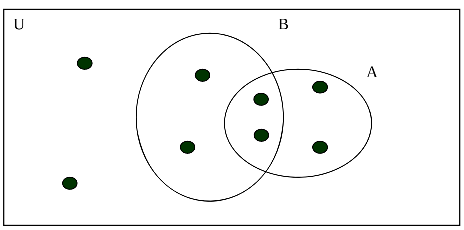
Example:

- A={1,2,3,6} B={4,7,8} Are these disjoint?
- Yes.
- $A \cap B = \emptyset$

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

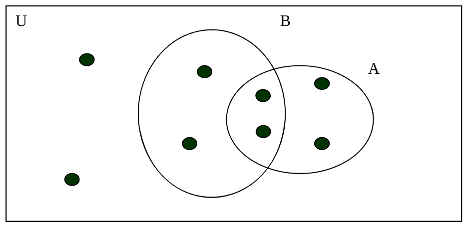


- Why this formula?

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

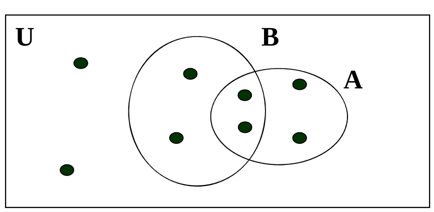


- Why this formula? Correct for an over-count.
- More general rule:
  - **The principle of inclusion and exclusion.**

Set difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \land x \notin B \}$.



Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}