Rectilinear Motion

Def: The motion of the particle along a straight line is said to be Rectilinear motion.

VELOCITY AND ACCELERATION IN RECTILINEAR MOTION

Let O be the fixed point and let x be the distance of the particle from O at any time t, Then

$$v = \frac{dx}{dt} \longrightarrow (1)$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \longrightarrow (2)$$

We can rewrite Eq. (2) as

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx} \longrightarrow (3)$$

MOTION WITH CONSTANT ACCELERATION

Let the particle be moving with constant acceleration along a straight line.

Let at time $t=0$, the particle be at the point O moving with a velocity u.

Integrating Eq. (2), we get

$$\frac{dx}{dt} = at + A$$

Using initial condition $v(0) = u$, we get

$$\boxed{u = A}$$

then

$$\frac{dx}{dt} = v = at + u \longrightarrow (4)$$

which gives velocity of the particle at any time t.

Now integrating Eq. (4), we get

$$x = \frac{1}{2}at^2 + ut + B \longrightarrow (5)$$

When $t=0$, the particle is at O. i.e $x=0$

Using this condition in (5), we get

$$0 = 0 + 0 + B$$

$$\Rightarrow \boxed{B = 0}$$

Therefore $x = \frac{1}{2}at^2 + ut$

$$\text{or } \boxed{x = ut + \frac{1}{2}at^2} \longrightarrow (6)$$

distance travelled by particle in time t.

<u>Example</u> Find the distance travelled by a particle moving in a straight line with uniform acceleration, in the nth unit of time.

<u>Solution</u>

Let x_1 and x_2 be the distance travelled by the particle in the first n and $n-1$ units of time respectively.

then

$$x = ut + \frac{1}{2}at^2$$

gives

$$x_1 = un + \frac{1}{2}an^2$$

and $$x_2 = u(n-1) + \frac{1}{2}a(n-1)^2$$

So distance travelled in nth unit of time is

$$x_1 - x_2 = u(n-n+1) + \frac{1}{2}(an^2 - an^2 - a + 2an)$$

$$= u + \frac{1}{2}(2an - a)$$

$$\boxed{x_1 - x_2 = u + \frac{a}{2}(2n-1)}$$

SOLUTION OF EQ. 2, UNDER OTHER B.Cs.

Now integrating Eq. 2,

$$\frac{dx}{dt} = at + A \longrightarrow ①^*$$

Using the conditions

$$t=0, \quad x=0;$$
$$t=t_1, \quad x=x_1$$

In this case, we can't find A.

Integrating again

$$x = \frac{at^2}{2} + At + B \longrightarrow ②^*$$

Now Applying B.C.

$t = 0 \; , \; x = 0$

$0 = 0 + 0 + B$

$\Rightarrow \boxed{B = 0}$

and $t = t_1 \; , \; x = x_1$

$x_1 = \frac{1}{2} a t_1^2 + A t_1$

$\Rightarrow A t_1 = x_1 - \frac{1}{2} a t_1^2$

$A = \frac{x_1}{t_1} - \frac{1}{2} \frac{a t_1^2}{t_1}$

$$\boxed{A = \frac{x_1}{t_1} - \frac{1}{2} a t_1}$$

So (2) becomes

$$x = \frac{1}{2} t^2 + \left(\frac{x_1}{t_1} - \frac{1}{2} a t_1 \right) t \quad \text{———} \quad (3)^*$$

Eq.: $(3)^*$ gives distance of the particle at any time, from the point O.

Comparing Eq. $(3)^*$ & (6) we can say that $\frac{x_1}{t_1} - \frac{1}{2} a t_1$ is the initial velocity of the particle at $t = 0$.

SOLUTION OF EQ. $a = v\frac{dv}{dx}$

This eq. can be written as

$$a \cdot dx = v\,dv$$

Integrating, we get

$$ax = \frac{v^2}{2} + C$$

Using condition $v = u$ when $x = 0$ in above eq., we get

$$0 = \frac{u^2}{2} + C \Rightarrow \boxed{C = -\frac{u^2}{2}}$$

$$\therefore \quad ax = \frac{v^2}{2} - \frac{u^2}{2}$$

$$\text{or} \quad \boxed{v^2 - u^2 = 2ax}$$

MOTION WITH VARIABLE ACCELERATION

We will consider different cases

i) TIME - DEPENDENT - ACCELERATION

When acceleration is function of time only we can write

$$a(t) = \frac{d^2x}{dt^2}$$

Integrating, we get

$$\frac{dx}{dt} = \int a(t)\,dt + A = b(t) + A$$

Integrating again, we get

$$x = \int b(t)\,dt + At + B$$

where A and B are constants of integration.

<u>Example</u> Find the distance travelled and velocity attained by a particle moving in a straight line, at any time t if it starts from rest at $t=0$ and is subject to an acceleration $t^2 + \sin t + e^t$.

<u>SOLUTION</u>

$$\frac{d^2x}{dt^2} = t^2 + \sin t + e^t$$

Integrating, we get

$$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t + A$$

when $t=0$, $\frac{dx}{dt} = 0$ so

$$0 = 0 - 1 + 1 + A \Rightarrow \boxed{A = 0}$$

So $$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t$$

Integrating again, we get

$$x = \frac{t^4}{12} - \sin t + e^t + B$$

When $t=0$, $x=0$

$$\therefore \quad 0 = 0 - 0 + 1 + B \Rightarrow \boxed{B = -1}$$

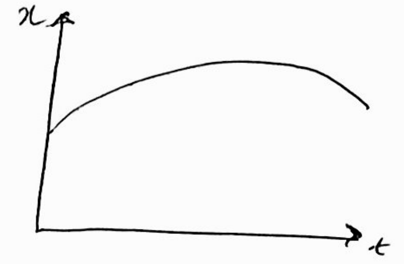
Thus $$x = \frac{t^4}{12} - \sin t + e^t - 1$$

Do other two cases from book.

<u>Note</u> velocity of the particle is given by

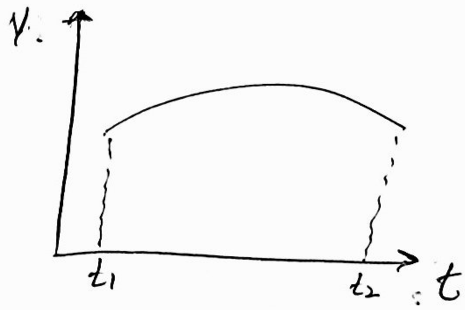
$$v = \frac{dx}{dt}.$$

$\frac{dx}{dt}$ is slope of the tangent to the curve $x = x(t)$ in tx-plane at a point (t, x). the curve is known as Space - time curve.



Thus velocity of particle at a distance x is equal to slope of the tangent to the space - time curve at a point whose ordinate is x.

If velocity is constant, the slope of space time curve remains same. i.e curve is straight line.



Similarly for acceleration

$$a = \frac{dv}{dt}$$

Also

$$v = \frac{dx}{dt}$$

Velocity - time curve

$$\int_{t_1}^{t_2} v\,dt = x \Big|_{t_1}^{t_2} = x(t_2) - x(t_1)$$

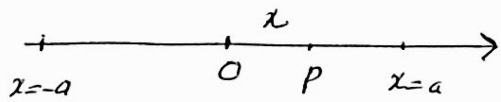
where $x(t_2) - x(t_1)$ is the distance travelled by the particle in the interval (t_1, t_2)

Hence. Slope of the velocity - time curve of a particle moving in a straight line gives its acceleration and the area under the curve gives the distance travelled by the particle.

SIMPLE HARMONIC MOTION

It is the motion of the particle moving in a straight line with an acceleration which is always directed towards a fixed point in the line and is propotional to the distance of the particle from that point.

Let O be the origin.



Let particle be at point P distance x from O as shown in fig.

Then

$$a = \frac{d^2x}{dt^2}$$

For simple harmonic motion this acceleration is propotional to x and directed towards O. So

$$\frac{d^2x}{dt^2} = -\lambda x \longrightarrow (1)$$

where λ is constant of popotionality.

When the particle moves away from O, acceleration is acting against it so that as time progsesses, velocity becomes lesser and lesser.

Assume that velocity vanishes at $x=a$.

Also let at $t=0$, particle be at point $x=a$

Multiplying Eq. (1) by $2\frac{dx}{dt}$.

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\lambda x\, 2\frac{dx}{dt}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -\lambda \frac{d}{dt}(x^2)$$

Integrating we get

$$\left(\frac{dx}{dt}\right)^2 = -\lambda x^2 + A.$$

$\frac{dx}{dt} = 0$ when $x = a$, so that

$$0 = -\lambda a^2 + A \Rightarrow \boxed{A = \lambda a^2}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -\lambda x^2 + \lambda a^2$$

$$= \lambda(a^2 - x^2)$$

$$\frac{dx}{dt} = \pm\sqrt{\lambda(a^2-x^2)} \longrightarrow (2)$$

Case I $\quad \frac{dx}{dt} = \sqrt{\lambda(a^2-x^2)}$

or $\quad \frac{dx}{\sqrt{(a^2-x^2)}} = \sqrt{\lambda}\, dt$

Integrating we get

$$\sin^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + B \quad\text{———}\quad 1^*$$

At $t = 0$, $x = a$

$$\sin^{-1}\frac{a}{a} = \sqrt{\lambda}(0) + B$$

$$\Rightarrow \boxed{B = \pi/2}$$

Thus

$$\sin^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + \pi/2$$

$$\frac{x}{a} = \sin(\sqrt{\lambda}\, t + \pi/2)$$

or $$x = a \sin(\sqrt{\lambda}\, t + \pi/2)$$

or $$\boxed{x = a \cos\sqrt{\lambda}\, t} \longrightarrow (3)$$

<u>Case II</u>

$$\frac{dx}{dt} = -\sqrt{\lambda(a^2-x^2)}$$

or $$\frac{dx}{\sqrt{x^2-a^2}} = \sqrt{\lambda}\, dt$$

Integrating, we get

$$\cos^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + C$$

When $t=0$, $x=a$.

$$\Rightarrow \quad C = 0$$

So $$\boxed{x = a \cos\sqrt{\lambda}\, t}$$

Same as Eq. (3).

<u>Note</u> Eq. of motion will be different under different B.Cs. e.g if particle is at point O when we start measuring time, then

$x = a \sin\sqrt{\lambda}\, t$ (At $t=0, x=0$)

Constt. B in 1* will be zero in This case

GENERAL SOLUTION OF $\frac{d^2x}{dt^2} = -\lambda x$

$$\frac{d^2x}{dt^2} = -\lambda x.$$

Auxilliary eq. is

$$m^2 + \lambda = 0 \Rightarrow m^2 = -\lambda \Rightarrow m = \pm i\sqrt{\lambda}$$

$$\boxed{x = \cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t}$$

NATURE OF SIMPLE HARMONIC MOTION

Simiple harmonic motion is described by

$$x = a\cos\sqrt{\lambda}t. \longrightarrow (1)$$

1. Here x cannot be greater that a because $\cos\sqrt{\lambda}t$ cannot exceed unity. So particle is bound to stay with in a distance a from O. This length a is called amplitude of motion and O is centre of motion.

2. Since $\frac{dx}{dt} = \pm\sqrt{\lambda(a^2 - x^2)}$

$$\max v = \sqrt{\lambda}\,a \quad \text{at} \quad O$$

and $\quad v = 0 \quad$ at $\quad x = \pm a$

The motion of the particle from $x = a$ to $x = -a$ and then again from $x = -a$ to $x = a$ completes one oscillation or vibration.

3. Eq. of motion is

$$x = a \cos \sqrt{\lambda} t$$

We can write

$$x = a \cos (\sqrt{\lambda} t + 2\pi)$$

$$x = a \cos \sqrt{\lambda} \left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

implies that distance of the particle at any time t is same as at another time $t + \frac{2\pi}{\sqrt{\lambda}}$.

Also $\frac{dx}{dt} = -a \sqrt{\lambda} \sin \sqrt{\lambda} t$

remain unaltered if t is increased by $t + \frac{2\pi}{\sqrt{\lambda}}$

This quantity $T = \frac{2\pi}{\sqrt{\lambda}}$ is called <u>time period of oscillation</u>

4. The number of oscillations which the particle completes in a unit of time is known as <u>frequency of oscillation</u>.