1. THEORY OF THE FINITE ELEMENT METHOD WITH ONE-DIMENSIONAL EXAMPLES

13-10-86

1.1 SOLUTION OF BOUNDARY VALUE PROBLEMS

A boundary value problem is one which is governed by one or more differential or integral equations within a specified domain, and by boundary conditions on the periphery of that domain. The solution may be obtained by extremising a functional, or a set of functionals, over the whole domain.

As an example, structural analysis problems are governed by the following equations:

(i) Compatability, or strain-displacement relations.

(ii) Stress-strain relations.

(iii) Equilibrium equations.

The solution may be obtained by minimising the total potential energy of the structure.

Generally speaking, in order to solve a boundary-value problem, one of the following methods may be used.

a) Exact or Closed-Form Solution

The first step in attempting to solve any problem is to try to find an exact solution. The basic methods for achieving this are:

(i) Direct integration

(ii) Separation of variables

(iii) Laplace transform

(iv) Infinite series, etc

b) Approximate Solutions Tried, just

If the problem is too difficult to be solved exactly, an acceptable approximate solution may be found. There are many useful approximate methods, which can be reviewed in Reference 1.1. The general approximate methods are summarised as follows.

(i) The Finite Difference Method is an approximate method for solving partial differential equations. It has been used to solve a wide range of problems. These include linear and non-linear, time dependent and independent problems

The problem domain is to be discretised pointwise and the value of the field function at any point is to be determined in terms of its values at some of the surrounding points. The basic finite difference algorithms for the solution of differential equations can be reviewed in Reference 1.2. The method has been used successfully for the solution of some engineering problems. The basic difficulties encountered in using such a method are as follows:

1. Iterative schemes may be required, their numerical stability, depending upon many parameters, may require some experience if convergence is to be achieved.
2. It is difficult to deal with complex boundary conditions.
3. The solution algorithm depends upon the equation type; parabolic, elliptic, or hyperbolic, and it is difficult to generalise.

(ii) Trial Function Methods is one of the powerful methods of constructing exact solutions

(A solution is assumed in terms of unknown parameters, for a 2-D problem:)

$$\tilde{u}(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + \ldots$$

To obtain such unknown parameters:

1. Extremise a functional over the whole domain. eg.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The solution is that which satisfies

$$\chi = \iint_{Domain} \left(\left(\frac{\partial \tilde{u}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{u}}{\partial y} \right)^2 \right) dxdy = \text{extremum}$$

Hence,

$$\frac{\partial \chi}{\partial a_i} = 0 \qquad i = 1, 2, ..$$

This procedure is known as the Rayleigh-Ritz method.
It is a numerical method of finding approximations to equations that are difficult to solve analytically. It is a approximate method which is used to find displacement function of an object. It is also known as variational approach. It is an Integral approach method.

2. Minimise the weighted error obtained by substituting the assumed equation in the differential equation.

For $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\iint_{Domain} w_i(x,y) \left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} \right) dxdy = 0$$

$i = i, 2, ..$

This procedure is known as the Weighted-Residual method.
It is used to find or solve the ordinary differential equation

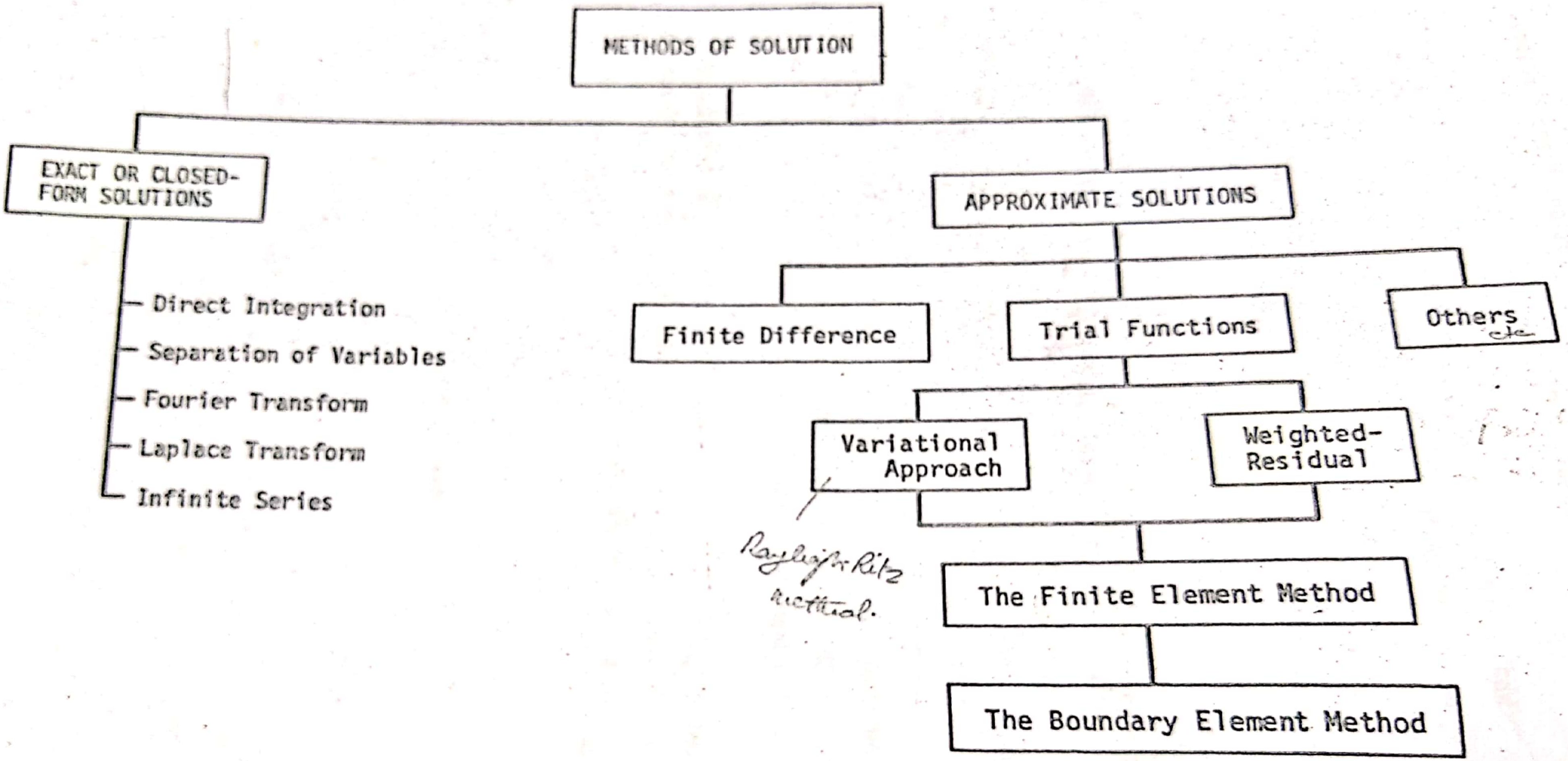
The basic disadvantages of the trial function methods are as follows:

1. Assuming a single solution, valid for the whole domain, may require an excessive number of terms which may, in turn, lead to high rounding-off errors.

2. It is difficult to satisfy general boundary conditions.

(iii) The Finite Element Method

This method is based upon trial function methods once their difficulties have been overcome as follows



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1.2 THE VARIATIONAL APPROACH

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Calculus of variation is a classical mathmematical subject, which is mainly concerned with functional extremisation problems. For such problems, it is required to find the unknown function, or functions which extremise a functional or a system of functionals, satisfying some given boundary conditions. For the case of one-dimensional problems, the variational statement can generally be expressed as follows Given

Q

$$\chi(y) = \int_{x_1}^{x_2} F(x, y, y', y'', ..) \, dx = \text{extremum.}$$

In order to extremise the functional χ, its first variation should vanish, i.e.

$$\delta \chi = 0$$

This condition has an associated equivalent differential equation called the Euler, or Euler-Lagrange, equation, which can be expressed as follows:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - .. = 0$$

In 1870, Rayleigh presented a direct method for the solution of variational problems and Ritz refined and extended Rayleigh's method in 1909. The Rayleigh-Ritz method can also be used for the solution of a differential equation, provided that an equivalent variational statement can be found. More details about calculus of variation can be reviewed in the specialised texts such as Reference 1.6.

Example

1.2.1 Steps of the Rayleigh-Ritz Solution

Consider that it is required to extremise the following functional

Given

$$\chi = \int_0^1 \left[\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + xy\right] dx \; \chi.$$

with the boundary conditions

$$y(0) = 0$$
$$y(1) = 0$$

find $y(x) = ?$

Note that the corresponding differential equation can be deduced from the Euler-Lagrange equation as follows

$$\chi = \int_0^1 (\tfrac{1}{2} y'^2 + xy)\, dx$$

$$F(x, y, y') = \tfrac{1}{2} y'^2 + xy$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = x - \frac{d}{dx} y' = 0$$

i.e. $$\frac{d^2y}{dx^2} - x = 0$$

<u>Step 1</u>

Assume an approximate solution of the form

$$y \simeq \tilde{y} = \sum_{j=1}^{m} \alpha_j \, \psi_j(x)$$

α_j are coefficients to be determined,

ψ_j are elements of a given sequence of a linearly independent function. (The basis function)

The basis function can be a trigonometric series, Legendre polynominals, etc. The simplest choice is the algebraic series,

$$1, \; x, \; x^2, \; ..$$

i.e. $$\psi_j(x) = x^{(j-1)}$$

Step 2

Verify that the assumed solution satisfies the given boundary conditions.

The given boundary conditions (say l conditions) are to be substituted into the previous solution and l coefficients are to be eliminated to reduce the solution into the form

$$\tilde{y} = \sum_{j=1}^{n} a_j \ \phi_j (x)$$

where $n = m-l$

For the given example

$y(0) = 0$
$y(1) = 0$

$$y(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \ \cdots\cdots$$

$$y(0) = 0 = \alpha_1$$

$$y(1) = 0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots\cdots$$

i.e.

$$\alpha_1 = 0$$
$$\alpha_2 = -(\alpha_3 + \alpha_4 + \cdots\cdots)$$

$$y(x) = -(\alpha_3 + \alpha_4 + \cdots\cdots) x + \alpha_3 x^2 + \alpha_4 x^3 + \cdots\cdots$$

$\psi_j = x^{i+1} - x$

$$= a_1 \ \phi_1(x) + a_2 \ \phi_2(x) + a_3 \ \phi_3(x) + \cdots\cdots$$

where $\quad a_1 = \alpha_3, \quad a_2 = \alpha_4, \ \cdots\cdots$

$$\phi_j(x) = x^{(j+1)} - x$$

Step 3

Substitute the approximate solution into the variational expression, to obtain

$$\chi \ (a_1, \ a_2, \ \ldots, \ a_n) = \int_{\text{domain}} F \ (\tilde{y}) \ dx$$

For the given example

$$F(y) = \tfrac{1}{2}(y')^2 + xy$$

$$y(x) = a_1(x^2-x) + a_2(x^3-x) + \ldots.$$

$$\frac{dy}{dx} = a_1(2x-1) + a_2(3x^2-1) + \ldots.$$

$$F(y) = \tfrac{1}{2}(a_1(2x-1) + a_2(3x^2-1) + \ldots.)^2 + (a_1(x^3-x^2) + a_2(x^4-x^2) + \ldots.)$$

$$\chi = \int_0^1 F(\bar{y})\,dx$$

$$= \tfrac{1}{2}\int_0^1 (a_1(2x-1) + a_2(3x^2-1) + \ldots.)^2\,dx + \int_0^1 (a_1(x^3-x^2) + a_2(x^4-x^2) + \ldots.)\,dx$$

Step 4

Extremise the variational functional

From variational calculus, in order to extremise χ,

$$\delta\chi = 0$$

i.e. $$\frac{\partial\chi}{\partial a_1}\delta a_1 + \frac{\partial\chi}{\partial a_2}\delta a_2 + \ldots. + \frac{\partial\chi}{\partial a_n}\delta a_n = 0$$

or $$\frac{\partial\chi}{\partial a_i} = 0, \quad i = 1, 2, .., n$$

For the given example

(i) $$\frac{\partial\chi}{\partial a_1} = \tfrac{1}{2}\cdot 2\int_0^1 (2x-1)\cdot(a_1(2x-1) + a_2(3x^2-1) + \ldots.)\,dx + \int_0^1 (x^3-x^2)\,dx = 0$$

Step 5

Solve the resulting equations

$$\underline{C}_{nxn}\ \underline{a}_{nx1} = \underline{b}_{nx1}$$

where

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$$\underline{a} = \{ a_1 \quad a_2 \quad .. \quad a_n \}$$

$$\underline{b} = \{ b_1 \quad b_2 \quad .. \quad b_n \}$$

If it is required to complete the solution for the case of

$$y = a_1 (x^2-x) + a_2(x^3-x)$$

the coefficients of the above system of equations can be expressed as follows:

$$C_{11} = \int_0^1 (2x-1)^2 \ dx$$

$$= \frac{(2x-1)^3}{2 \cdot 3} \Big|_0^1 = \frac{1}{3}$$

$$C_{12} = \int_0^1 (2x-1)\ (3x^2-1)\ dx$$

$$= \int_0^1 (6x^3 - 3x^2 - 2x + 1)\ dx$$

$$= \frac{6}{4} - \frac{3}{3} - \frac{2}{2} + 1 = \frac{1}{2} = C_{21}$$

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$$C_{22} = \int_0^1 (3x^2-1)^2\, dx$$

$$= \int_0^1 (9x^4 - 6x^2 + 1)\, dx$$

$$= \frac{9}{5} - \frac{6}{3} + 1 = \frac{4}{5}$$

$$b_1 = -\int_0^1 (x^3-x^2)\, dx$$

$$= -\left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{12}$$

$$b_2 = -\int_0^1 (x^4-x^2)\, dx$$

$$= -\left(\frac{1}{5} - \frac{1}{3}\right) = \frac{2}{15}$$

Hence By solving we get

$$\frac{1}{3} a_1 + \frac{1}{2} a_2 = \frac{1}{12} \qquad ..(1)$$

$$\frac{1}{2} a_1 + \frac{4}{5} a_2 = \frac{2}{15} \qquad ..(2)$$

multiply (1) by 12 and (2) by 30

$$4a_1 + 6a_2 = 1 \qquad ..(1)'$$

$$15a_1 + 24a_2 = 4 \qquad ..(2)'$$

$$a_1 = \frac{\begin{vmatrix} 1 & 6 \\ 4 & 24 \end{vmatrix}}{\begin{vmatrix} 4 & 6 \\ 15 & 24 \end{vmatrix}} = \frac{24 - 24}{96 - 90} = \frac{0}{6} = 0$$

$$a_2 = \frac{\begin{vmatrix} 4 & 1 \\ 15 & 4 \end{vmatrix}}{6} = \frac{16 - 15}{6} = \frac{1}{6}$$

$y(x) = a_1(x^2-x) + a_2(x^3-x)$

b) Variational Statement

For many problems of continuum mechanics there are forms of energy balance theorems, which provide variational statements directly. There are many energy theorems which can be used for structural analysis, as shown by Reference 1.8. The basic energy theorem employed in this text is the minimum total potential energy theorem, which states that

The exact solution of a structural analysis problem is the one, from all compatible displacement fields, which makes the total potential energy of the structure a minimum.

The total potential energy of the structure can be expressed as follows

$$\chi = U + V$$

where

U = the strain energy of the structure, defined as follows.

$$U = \iiint_{\text{structure}} \left(\int \underline{\sigma}^t d\,\underline{\varepsilon} \right) d(\text{Vol.})$$

and for a linear elastic material,

$$U = \tfrac{1}{2} \iiint_{\text{structure}} \underline{\sigma}^t\, \underline{\varepsilon}\; d(\text{Vol.})$$

V is the potential energy of the external loads

= -W, where W is the work done by the external loads.

c) Example

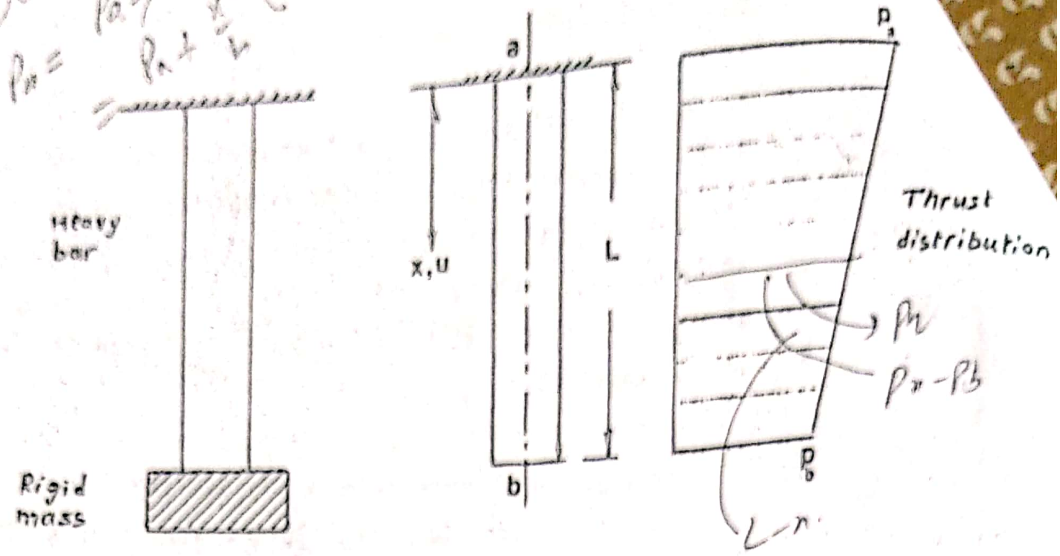
In order to illustrate the basic concepts, a simple structural problem will be solved.

$1 - \frac{x}{L} = \frac{P_x}{P_a - P_b}$

$(1 - \frac{x}{L})(P_a - P_b) = P_x$

$P_x = P_a - P_b - \frac{P_a x}{L}$

$P_x + \frac{x}{L}(P_b - P_a)$



Suppose that it is required to determine the deformation and stress-strain distribution for a heavy bar which is hung vertically from one end, and carries a heavy rigid mass at the other. The problem can be modelled as a bar ab fixed at end 'a' and subjected to a linearly-distributed thrust.

$$\frac{L-x}{L} = \frac{P_x - P_b}{P_a - P_b}$$

$$P(x) = P_a + \frac{x}{L}(P_b - P_a)$$

Let

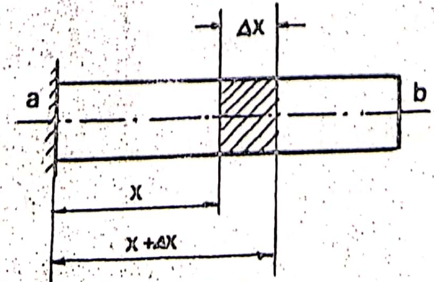
A = Cross-sectional area of the bar

E = Young's modulus of the material

(i) Stresses and Strains

The stress at any point x can be defined as follows

$$\sigma = \frac{\text{Thrust Force}}{\text{Cross-sectional area}} = \frac{P(x)}{A}$$



Let the displacement at any point of co-ordinate x, be u, then at x + Δx the displacement is

$$u + \frac{du}{dx}\,\Delta x \; + \text{higher order terms}$$

Hence the engineering strain can be defined as follows

$$\text{Engineering strain} \;=\; \lim_{\Delta x \to 0} \frac{\text{Change of length}}{\text{Original length}}$$

higher order

$$= \lim_{\Delta x \to 0} \frac{\left(u + \frac{du}{dx}\Delta x + \frac{1}{2}\frac{d^2u}{dx^2}(\Delta x)^2 + ..\right) - u}{\Delta x}$$

i.e.

$$\varepsilon = \frac{du}{dx} = \frac{\delta u}{\delta x}$$

The stress can be expressed in terms of the displacement by employing Hooke's law

$$\sigma = E\varepsilon = E\,\frac{du}{dx}$$

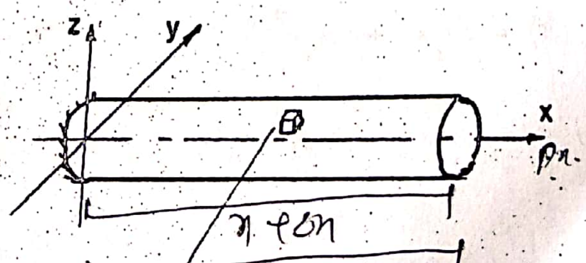
(iii) Energy or Variational Expression

Strain energy

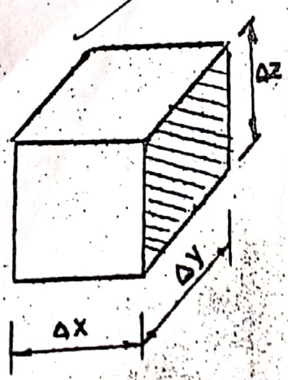
$$U = \frac{1}{2}\iiint \sigma^t \varepsilon \;\; dxdydz$$

$$= \frac{1}{2}\iiint E\left(\frac{du}{dx}\right)^2 \; dxdydz$$

Since u is function of x only, then



Element of Volume L.



Element of volume

$$U = \frac{1}{2} \int E \left(\frac{du}{dx}\right)^2 \left(\iint dydz\right) dx$$

From the above figure, it can be shown that

$$\iint dydz = A$$

i.e. $$U = \frac{1}{2} \int_0^L AE \left(\frac{du}{dx}\right)^2 dx$$

The work done by the external force can be expressed as follows

$$W = \int_{Bar} Pdu = \int_0^L P \left(\frac{du}{dx}\right) dx$$

Finally, the total potential energy is given by

$$\boxed{X = U - W = \frac{1}{2} \int_0^L AE \left(\frac{du}{dx}\right)^2 dx - \int_0^L P \left(\frac{du}{dx}\right) dx}$$ integral equation

(iii) Rayleigh-Ritz Solution

Step 1

Assume a solution of the following form

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

Step 2

Verify that the assumed solution satisfies the given boundary condition.

At $x = 0$ $u = 0$

thus $\alpha_1 = 0$

$$u(x) = \alpha_2 x + \alpha_3 x^2$$

$$U(x) = a_1 x + a_2 x^2$$

dividing the body into an equivalent load

Step 3

Substitute the approximate solution into the variational expression.

$$u(x) = a_1 x + a_2 x^2$$

$$\frac{du}{dx} = a_1 + 2\, a_2 x$$

$$\chi = \frac{1}{2} \int_0^L AE \left(\frac{du}{dx}\right)^2 dx - \int_0^L P \left(\frac{du}{dx}\right) dx$$

$$= \frac{1}{2} \int_0^L AE \; (a_1 + 2a_2 x)^2 \; dx$$

$$- \int_0^L \left(P_a + \frac{x}{L} (P_b - P_a) \right) (a_1 + 2a_2 x) \; dx$$

Step 4

Extremise the variational functional

$$\frac{\partial \chi}{\partial a_1} = \frac{\partial \chi}{\partial a_2} = 0$$

$$\frac{\partial \chi}{\partial a_1} = \int_0^L AE \; (a_1 + 2a_2 x) \; dx$$

$$- \int_0^L \left(P_a + \frac{x}{L} (P_b - P_a)\right) dx = 0$$

$$AE \; (a_1 x + a_2 x^2) - \left(P_a x + \frac{x^2}{2L} (P_b - P_a) \right) \Big|_0^L = 0$$

$P_a l + \frac{l^2}{2l}$

or

$$a_1 + La_2 = \frac{1}{2AE} \; (P_a + P_b) \quad \ldots \; (1)$$

$$\frac{\partial X}{\partial a_2} = \int_0^L 2\,x\,AE\,(a_1 + 2a_2x)\,dx$$

$$- \int_0^L 2\,x\,(\,P_a + \frac{x}{L}\,(P_b - P_a)\,)\,dx = 0$$

$$\int_0^L AE\,(a_1\,x + 2a_2x^2)\,dx$$

$$- \int_0^L (x\,P_a + \frac{x^2}{L}\,(P_b - P_a))\,dx = 0$$

$$AE\,(a_1\,\frac{x^2}{2} + \frac{2a_2x^3}{3}) - (\,\frac{x^2}{2}\,P_a + \frac{x^3}{3L}\,(P_b - P_a)\,)\Big|_0^L =$$

or

$$3a_1 + 4La_2 = \frac{P_a + 2P_b}{AE} \quad \ldots (2)$$

Step 5

Solve the resulting equations

$$a_1 + La_2 = \frac{1}{2AE}\,(P_a + P_b)$$

$$3a_1 + 4La_2 = \frac{1}{AE}\,(P_a + 2P_b)$$

Hence,

$$a_1 = \frac{P_a}{AE}$$

$$a_2 = \frac{P_b - P_a}{2LAE}$$

$u(x) = a_1(x) + a_2x^2$

and

$$u(x) = \frac{1}{AE}\,(\,x\,P_a + \frac{x^2}{2L}\,(P_b - P_a)\,)$$

Exercise 1.3

Solve the previous problem if the cross sectional area varies linearly along the bar length, i.e.

$$A(x) = \left(1 - \frac{x}{L}\right) A_a + \left(\frac{x}{L}\right) A_b$$

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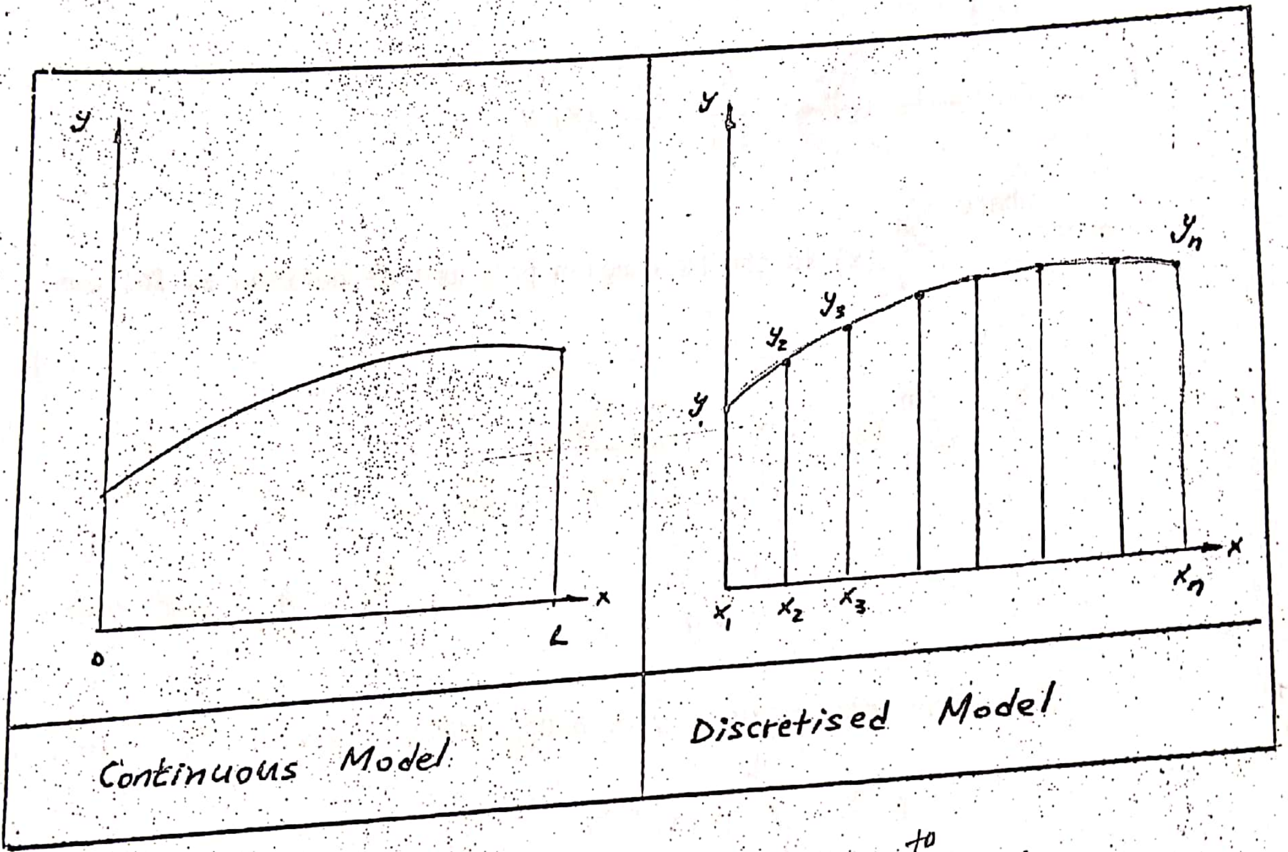
1.2.3 Discretisation Concepts

a) Pointwise Discretisation



Continuous Model　　Discretised Model

Let y(x) be a general field function in the domain (0, to L) , i.e.

$$0 \leq x \leq L$$

and it is required to calculate

$$I = \int_0^L y(x)\, dx$$

For exact graphical representation of $y(x)$, a theoretically infinite number of points is required. An acceptable approximation is to represent $y(x)$ in terms of a finite number of points, y_1, y_2, ..., y_n at $x = x_1, x_2, \ldots, x_n$. This concept is known as "Pointwise Discretisation".

An approximate continuous model can be obtained from the discretised model by fitting a continuous curve which passes through the points

$$(x_i, y_i) \quad , \quad i = 1, 2, .., n$$

This can be achieved generally by employing Lagrangian interpolation as follows

$$\tilde{y}(x) = \sum_{i=1}^{n} \mathcal{L}_i^n (x) \, y_i$$

where

$\mathcal{L}_i^n (x)$ is the Lagrangian polynominal defined as follows

$$\mathcal{L}_i^n (x) = \prod_{\substack{r=1, \\ r \neq i}}^{n} \left(\frac{x - x_r}{x_i - x_r} \right)$$

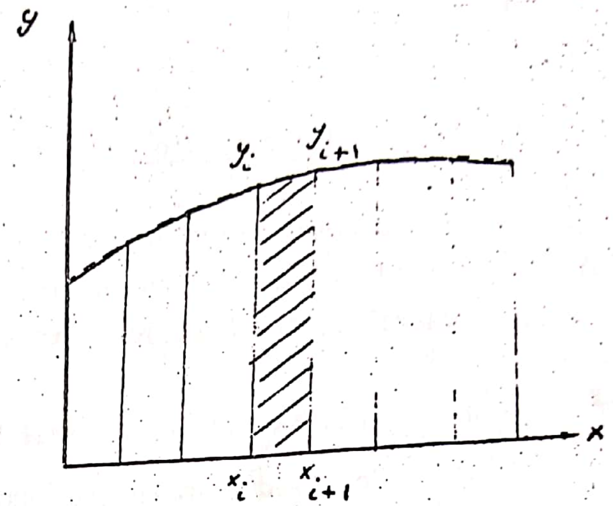
Hence the integration may be approximated analytically as follows

$$I \simeq \int_0^L \tilde{y}(x) \, dx$$

b) Piecewise Discretisation

Using pointwise discretisation, for the previous example, the function $y(x)$ can be approximated to a very high degree, but the resulting integral expression is not simple. The second approach is to divide the whole domain into a finite number of subdomains. The function $y(x)$

does not require to be approximated very accurately for each subdomain. An acceptable approximation is to use linear interpolation, as shown below:



For this case

$$\underbrace{\int y\,dx}_{\text{whole domain}} = \underbrace{\int_{x_1}^{x_2} y\,dx}_{\text{1st subdomain}} + \underbrace{\int_{x_2}^{x_3} y\,dx}_{\text{2nd subdomain}} + \dots$$

For the i^{th} subdomain

$$I_i = \int_{x_i}^{x_{i+1}} y(x)\,dx$$

$$\frac{y(x) - y_i}{y_{i+1} - y_i} = \frac{x - x_i}{x_{i+1} - x_i}$$

i.e. $$y(x) = y_i + \left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)(x - x_i)$$

It can be deduced that

$$I_i = (x_{i+1} - x_i)\ (y_{i+1} + y_i)\ /2$$

which is the well-known Trapezoidal Rule for numerical integration.

1.2.4 Discretisation of Rayleigh-Ritz Method

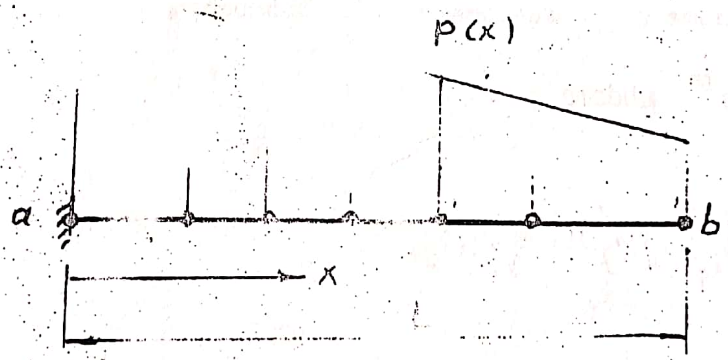
Any boundary value problem which has a variational statement can be solved by means of the Rayleigh-Ritz method as described earlier. Generally, the Rayleigh-Ritz procedure has two basic difficulties.

a) Satisfaction of general boundary conditions

b) The need to make the assumed solution valid for the whole domain.

The "Discretisation Concept" can be employed in order to overcome such difficulties.

a) Poi[illegible] [illegible]cre[illegible]



Consider the ba[illegible] [illegible]lem discussed before. The first step of the Rayleigh-Ritz [illegible] is to assume [illegible] [illegible]roximate solution as follows

$$u\,(x) = \sum_{j=1}^{m} \quad x \quad ^{-1}$$

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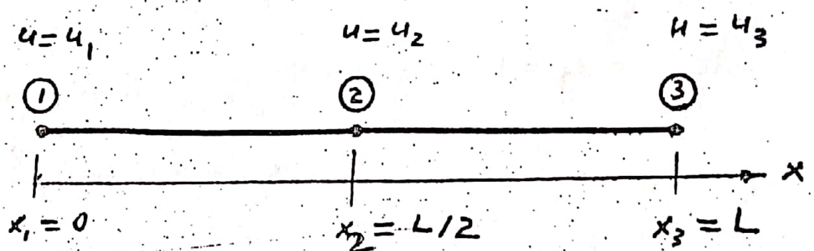
but it is difficult to allocate a physical meaning to the coefficients α_j. Alternatively, the values of u at m points; $u_1, u_2, \ldots, u_m$ can be assumed as unknown parameters, i.e.

$$u(x) = \sum_{j=1}^{m} u_j \; N_j(x)$$

$N_j(x)$, $j = 1, 2, \ldots, m$ are simple algebraic polynominals which satisfy the following condition

$$\boxed{N_j(x_i) = \delta_{ij}} \text{ (Kronecher Delta)}$$

Pointwise Discretisation of the Bar Example



Instead of assuming that

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad \ldots\ldots(a)$$

the following expression will be used

$$u(x) = u_1 N_1(x) + u_2 N_2(x) + u_3 N_3(x) \quad \ldots\ldots(b)$$

Applying Lagrange's Theorem, it can be shown that

$$N_1(x) = \frac{(x-x_2)\;(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2(x) = \frac{(x-x_1)\ (x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3(x) = \frac{(x-x_1)\ (x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Alternatively, to deduce (b) from (a), the following procedure can be employed

At $x = x_1 = 0$ $\quad u = u_1, \quad u_1 = \alpha_1$ ...(i)

At $x = x_2 = L/2$ $\quad u = u_2,$

$$u_2 = u_1 + \alpha_2 \frac{L}{2} + \alpha_3 \left(\frac{L}{2}\right)^2 \qquad \text{...(ii)}$$

At $x = x_3 = L$ $\quad u = u_3$

$$u_3 = u_1 + \alpha_2 L + \alpha_3 L^2 \qquad \text{...(iii)}$$

Multiply (ii) by 2

$$2u_2 = 2u_1 + \alpha_2 L + \alpha_3 \frac{L^2}{2} \qquad \text{(ii)'}$$

Subtract (ii)' from (iii)

$$u_3 - 2u_2 = -u_1 + \alpha_3 \frac{L^2}{2}$$

$$\alpha_3 = \frac{2}{L^2}(u_1 - 2u_2 + u_3)$$

and $$\alpha_2 L = u_3 - u_1 - \alpha_3 L^2$$

$$= u_3 - u_1 - 2u_1 + 4u_2 - 2u_3$$

$$= \qquad -3u_1 + 4u_2 - u_3$$

$$\alpha_2 = \frac{1}{L}(-3u_1 + 4u_2 - u_3)$$

Hence,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$= u_1 + \left(\frac{x}{L}\right)(-3u_1 + 4u_2 - u_3) + 2\left(\frac{x}{L}\right)^2 (u_1 - 2u_2 + u_3)$$

i.e.

$$u(x) = u_1 \left[1 - 3\left(\frac{x}{L}\right) + 2\left(\frac{x}{L}\right)^2\right]$$

$$+ u_2 \left[4\left(\frac{x}{L}\right) - 4\left(\frac{x}{L}\right)^2\right]$$

$$+ u_3 \left[-\left(\frac{x}{L}\right) + 2\left(\frac{x}{L}\right)^2\right]$$

and

$$N_1(x) = (1 - \xi)(1 - 2\xi)$$

$$N_2(x) = 4\xi(1 - \xi)$$

$$N_3(x) = \xi(2\xi - 1)$$

where

$$\xi = x/L$$

Points 1, 2, and 3 are called <u>nodes</u>

N_1, N_2 and N_3 are known as <u>shape functions</u>.

Example

$u=u_1$ ① $x_1=0$ — $u=u_2$ ② $x_2=\frac{L}{2}$ — $u=u_3$ ③ $x_3=L$

Rayleigh-Ritz Solution

Step 1

Assumed Solution.

$$u(\xi) = u_1 (1-\xi)(1-2\xi) + 4u_2\, \xi(1-\xi) + u_3\, \xi(2\xi - 1)$$

Step 2

B.C

$$u_1 = 0$$

i.e. $$u(\xi) = 4u_2\, \xi(1-\xi) + u_3\, \xi(2\xi - 1)$$

Step 3

Substitution

$$\chi = \chi(u_2, u_3)$$

Step 4

Extremisation.

$$\frac{\partial \chi}{\partial u_2} = 0$$

$$\frac{\partial \chi}{\partial u_3} = 0$$

Step 5

Solve the resulting equations.

It can be shown that

$$u_2 = \frac{L}{8AE} (3 P_a + P_b)$$

$$u_3 = \frac{L}{2AE} (P_a + P_b)$$

Exercise 1.4

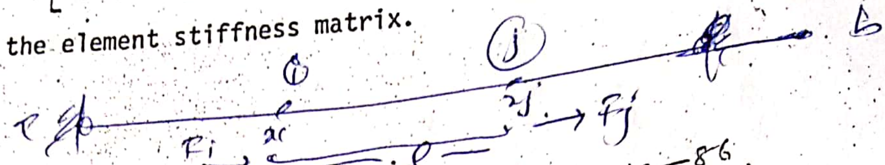
Use a 3-Point descretised Rayleigh-Ritz method to solve exercise 1.1.

b) Piecewise Discretisation

This process implies that the whole domain is to be divided into a number of subdomains, which are known as finite elements. Each subdomain should obey the same physical principle, applied to the whole domain, and can be treated as a separate problem. The equations for the whole domain can be obtained by summing the subdomain equations. The Rayleigh-Ritz method with both piecewise and pointwise discretisation, is the well-known finite element method, as illustrated below.

$$\underline{K}_{(e)} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is the element stiffness matrix.



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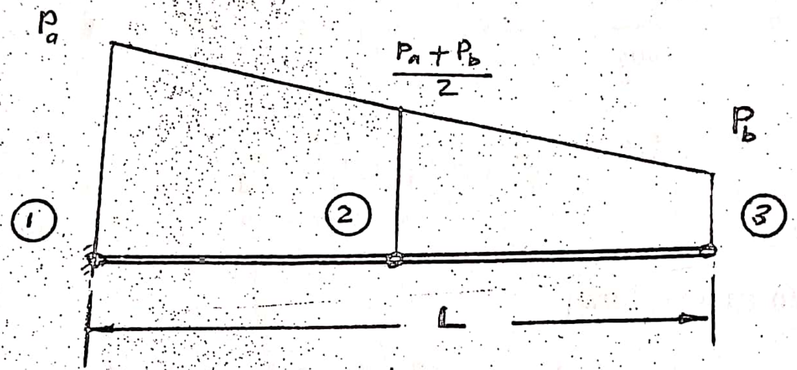
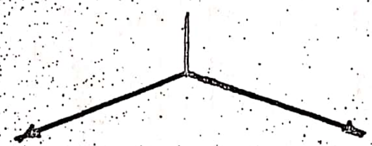
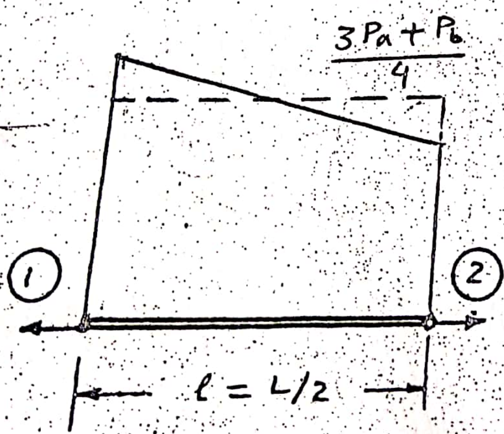
Steps of the Solution for the Whole Domain

Step 1 Discretisation of the Domain

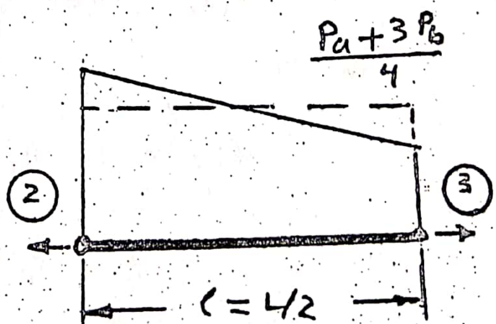
The domain can be divided into two subdomains and each subdomain can be considered a 2-node element, as shown below.







Element I



Element II

The equivalent nodal forces should satisfy the equilibrium equations. The logical solution is to assume that they are the average values of the thrust distribution for each element.

Step 2 Derivation of elements equations

The equations for the 2-node bar element have been derived as follows

$$\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} F_i \\ F_j \end{bmatrix}$$

Applying this to the previous problem, the following can be deduced,

For Element I

$$\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\ (3P_a + P_b)\ /4 \\ +\ (3P_a + P_b)\ /4 \end{bmatrix}$$

For Element II

$$\frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\ (P_a + 3P_b)\ /4 \\ +\ (P_a + 3P_b)\ /4 \end{bmatrix}$$

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Step 3 <u>Assembly of subdomain equations to obtain the equations for the whole domain</u>

It is useful to note, in the above element equations that

u_i is the total or actual displacement at the i^{th} node

F_i is a partial local force, for the considered subdomain.

Hence, the law of the assembly is based upon the mechanical principle.

The total force at each node = Σ local element forces at that node.

Assembly can be carried out by means of matrices, if the nodal displacement vector is the same for each element.

For the first element, inserting a correct number of zeros, the equations for the element can be expressed as follows.

$u_i - F_j$
$u_j - F_j$

$$\begin{bmatrix} (\frac{AE}{l}) & -(\frac{AE}{l}) & 0 \\ -(\frac{AE}{l}) & (\frac{AE}{l}) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -(3P_a + P_b)/4 \\ (3P_a + P_b)/4 \\ 0 \end{bmatrix}$$

Similarly for the second element,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & (\frac{AE}{l}) & -(\frac{AE}{l}) \\ 0 & -(\frac{AE}{l}) & (\frac{AE}{l}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -(P_a+3P_b)/4 \\ (P_a + 3P_b)/4 \end{bmatrix}$$

The two matrix equations can be added together as follows

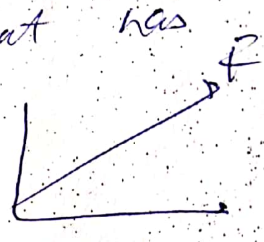
$$\begin{bmatrix} (\frac{AE}{l}) & -(\frac{AE}{l}) & 0 \\ -(\frac{AE}{l}) & 2(\frac{AE}{l}) & -(\frac{AE}{l}) \\ 0 & -(\frac{AE}{l}) & (\frac{AE}{l}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -(3P_a + P_b)/4 \\ (2P_a - 2P_b)/4 \\ (P_a + 3P_b)/4 \end{bmatrix}$$

or $\underline{K} \quad \underline{\delta} = \underline{F}$

where $\underline{K}$ is the structure stiffness matrix

$\underline{\delta}$ is the displacement vector, and

$\underline{F}$ is the force vector. , a vector that has with magnitude and direction



Step 4 <u>Apply the Boundary Conditions</u>

For this case, the boundary condition is $u_1 = 0$.

The first ~~equation~~ row and first column of $\underline{K}$ are to be eliminated, and a reduced system of equations will be obtained, as follows

$$2\left(\frac{AE}{l}\right) u_2 - \left(\frac{AE}{l}\right) u_3 = \frac{2P_a - 2P_b}{4}$$

$$-\left(\frac{AE}{l}\right) u_2 + \left(\frac{AE}{l}\right) u_3 = \frac{P_a + 3P_b}{4}$$

Step 5 Solve the resulting equations

$$2u_2 - u_3 = \frac{l}{AE}\left(\frac{2P_a - 2P_b}{4}\right) \qquad ..(1)$$

$$-u_2 + u_3 = \frac{l}{AE}\left(\frac{P_a + 3P_b}{4}\right) \qquad ..(2)$$

Adding the two equations,

$$u_2 = \frac{l}{AE}\left(\frac{3P_a + P_b}{4}\right)$$

and substituting into (2): $-\frac{l}{AE}\left(\frac{3P_a+P_b}{4}\right)+u_3=\frac{l}{AE}\left(\frac{P_a+3P_b}{4}\right)$

$$u_3 = \frac{l}{AE}(P_a + P_b)$$

From $l = L/2$,

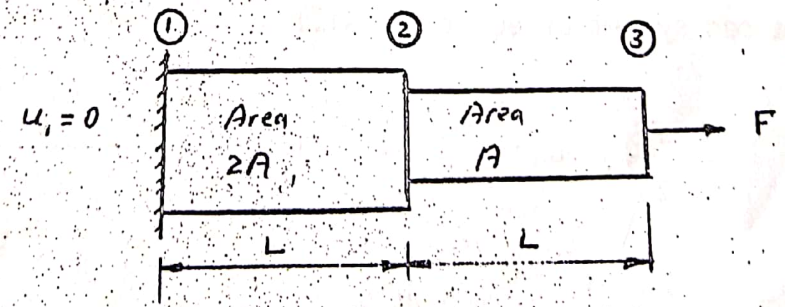
$$u_2 = \frac{L}{8AE}(3P_a + P_b)$$

$$u_3 = \frac{L}{2AE}(P_a + P_b)$$

as the exact solution.

Exercise 1.5

Using the FEM, solve the following problem.



Find u_2, u_3

and prove that the reaction at (1) = -F

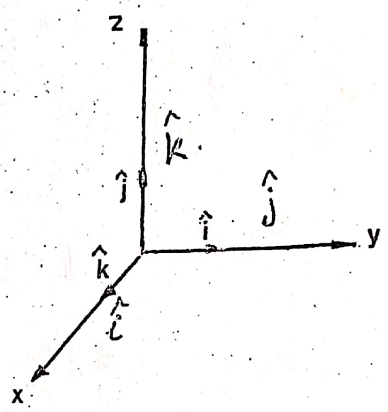
2. TWO-DIMENSIONAL FIELD PROBLEMS

27/10/85

2.1 INTRODUCTION

It is useful to review the previous concepts for problems in a multi-dimensional domain. A suitable example is the field problem which represents a significant class of physical problems.

Considering a three-dimensional domain, with reference axes ox, oy and oz, the following can be defined.



a) Directional Derivative

The directional derivative of any scalar function F(x, y, z) is Grad F or Napla F, which can be expressed as follows

$$\nabla F = \frac{\partial F}{\partial x}\hat{\imath} + \frac{\partial F}{\partial y}\hat{\jmath} + \frac{\partial F}{\partial z}\hat{k}$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along the x, y and z axes, respectively.

b) The Laplacian

The Laplacian of any function F is defined as

$$\nabla \cdot \nabla F = \nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

c) Steady-State Field Problems

The steady-state field problems can generally be described by means of the following differential equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial F}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial F}{\partial z}\right) + Q\,(x,\, y,\, z) = 0$$

The solution for such an equation is that which extremises the following variational statement

$$\chi = \frac{1}{2} \iiint \left(k_x \left(\frac{\partial F}{\partial x}\right)^2 + k_y \left(\frac{\partial F}{\partial y}\right)^2 + k_z \left(\frac{\partial F}{\partial z}\right)^2 \right) dxdydz$$

$$- \iiint \quad Q\,F\,dxdydz$$

Special Cases

i) Poisson's Differential Equation

If $k_x = k_y = k_z = k$ = constant, then

isotropic property

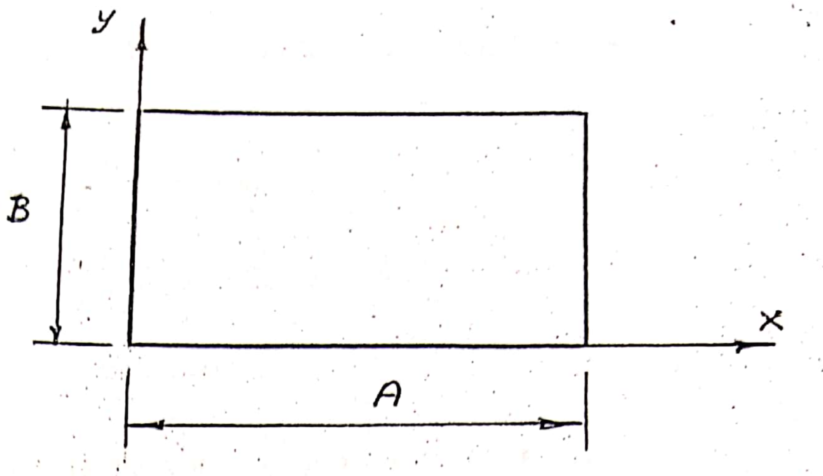
$$\nabla^2 F + Q/k = 0$$

$\nabla^2 F = 0$

$\nabla^2 F = Q/k$

which is known as Poisson's Differential Equation.

2.3 RAYLEIGH-RITZ SOLUTION



Consider a rectangular cross-section as shown in the above figure.

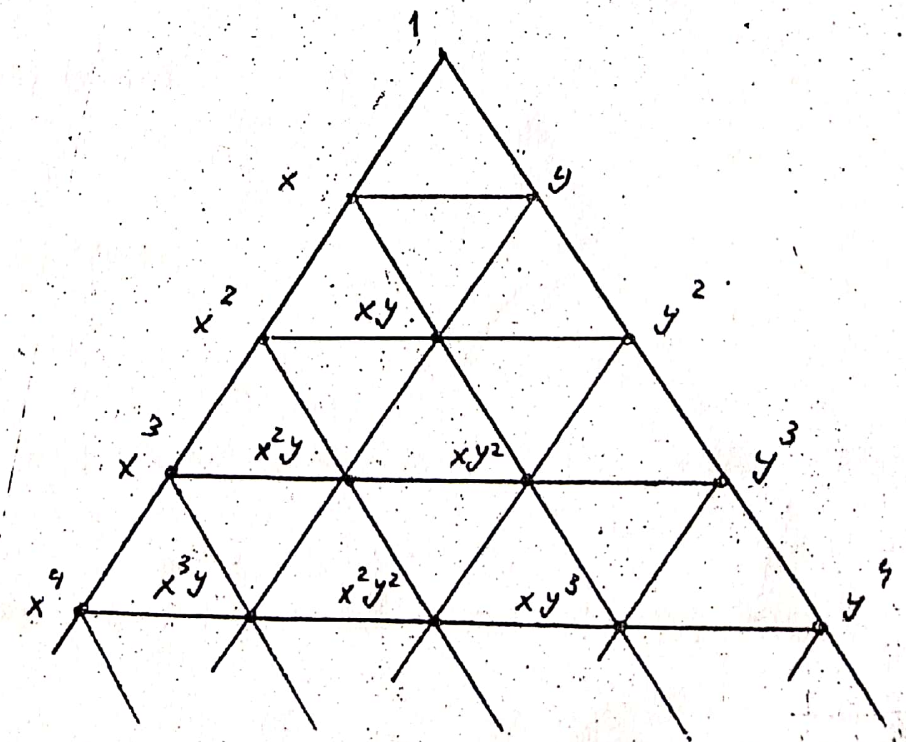
Step 1

Assume a solution

A solution can be expressed in terms of unknown parameters and algebraic polynominals as follows-

$$\omega(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \ldots$$

The terms of two-dimensional polynominals can be represented by the following Pascal's triangle



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Step 2

Verify that the assumed solution satisfies the given boundary conditions.

For this case, the boundary conditions are

$$\phi = 0 \quad \text{at} \quad x = 0, \; x = A, \; y = 0, \; y = B$$

An educated guess is to assume that,

$$\omega\,(x,y) = x(x-A)\;y(y-B)\;(\,a_1 + a_2 x + a_3 y + \ldots)$$

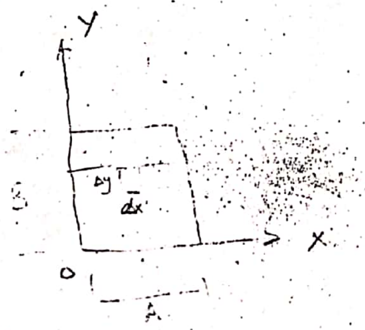
which satisfies the given boundary conditions.

Step 3

Substitute the assumed solution into the variational statement.

$$X_R = \frac{1}{2G}\int_0^B\int_0^A \left[\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right] dxdy - 2\int_0^B\int_0^A \omega\, dxdy$$

Consider a one-term solution, (for simplicity)



$$\omega = a_1\;x(x-A)\;y(y-B)$$

$$\frac{\partial\omega}{\partial x} = a_1\;(2x-A)\;(y^2-By)$$

$$\frac{\partial\omega}{\partial y} = a_1\;(x^2-Ax)\;(2y-B)$$

$$X_R = \frac{1}{2G}\int_0^B\int_0^A a_1^2\,[\,(2x-A)^2\,(y^4-2By^3+B^2y^2) + (x^4-2Ax^3+A^2x^2)$$

$$\cdot(2y-B)^2\,]\;dxdy - 2\int_0^B\int_0^A a_1\;(x^2-Ax)\;(y^2-By)\;dxdy$$

It can be deduced that

$$\chi_R = \frac{a_1^{\,2}}{2G}\left(\frac{A^3}{3}\,\frac{B^5}{30} + \frac{A^5}{30}\,\frac{B^3}{3}\right) - 2a_1\,\frac{A^3}{6}\,\frac{B^3}{6}$$

Step 4

Minimise the variational statement.

$$\frac{\partial \chi_R}{\partial a_1} = 0$$

$$\frac{a_1}{90G}\,A^3B^3\,(A^2 + B^2) - \frac{A^3B^3}{18} = 0$$

Step 5

Solve the resulting equation(s).

$$a_1 = \frac{5G}{A^2+B^2}$$

$w(x,y) = a_1\, x\,(x-A)(y)(y-B)$

i.e.

$$\omega\,(x,y) = \frac{5G}{A^2+B^2}\;x(x-A)\;\;y(y-B)$$

Shear Constant

$$T = \left(2\int_0^B\int_0^A \omega\,dxdy\right)\theta$$

$\Rightarrow \quad JG = 2\int_0^B\int_0^A w\,dxdy$

2.4 WEIGHTED-RESIDUAL SOLUTION

2.4.1 Common Steps

The following first three steps are the same for all of the weighted-residual methods.

Step 1

Assume a solution.

This is similar to Step 1 for the Rayleigh-Ritz solution.

$$\omega\,(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \ldots$$

Step 2

Verify that the assumed solution satisfies the given boundary conditions.

From the analysis employed for the Rayleigh-Ritz solution, it can be shown that

$$\omega\,(x,y) = x(x-A)\;y(y-B)\;(a_1 + a_2 x + a_3 y + \ldots)$$

satisfies the given boundary conditions.

Step 3

Substitute the assumed solution in the given differential equation, to define the residual function.

2.4.3 Integrated Least Squares

$$W_i(x,y) = \frac{\partial R(x,y)}{\partial a_i}$$

Hence,

$$W_1 = 2\,((x^2 - Ax) + (y^2 - By))$$

$$\int_0^B \int_0^A W_1\, R(x,y)\, dxdy = 0$$

which leads to

$$\int_0^B \int_0^A \{\, 4a_1\,((x^2 - Ax) + (y^2 - By))^2 + 4G\,((x^2 - Ax) + (y^2 - By))\,\} = 0$$

It can be deduced that

$$a_1 = \frac{15G(A^2+B^2)}{3A^4+5A^2B^2+3B^4}$$

2.4.4. Method of Moments

For this case

$$W_i(x,y) = 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ \ldots$$

For one-term solution, $W_1 = 1$

$$\int_0^B \int_0^A \{\ 2a_1\,((x^2 - Ax) + (y^2 - By)) + 2\,G\,\}\, dxdy = 0$$

$$2a_1 \left(\left(\frac{A^3}{3} - \frac{A^3}{2} \right) B + \left(\frac{B^3}{3} - \frac{B^3}{2} \right) A \right) + 2\,G\,AB = 0$$

Hence,

$$a_1 = \frac{6G}{A^2+B^2}$$

2.4.5 Galerkin Method

For the case of Galerkin's method,

$$\iint_{\text{domain}} \phi_i(x,y)\; R(x,y)\; dxdy = 0$$

$$\phi_1 = x(x - A)\; y(y - B)$$

$$\phi_2 = x\,\phi_1$$

$$\phi_3 = y\phi_1\;, \ldots \text{etc}$$

For the case of one-term solution:

$$\int_0^B \int_0^A (x^2 - Ax)\,(y^2 - By)\,\{ 2a_1 (x^2 - Ax + y^2 - By) + 2\,G\}\; dxdy = 0$$

Hence, it can be deduced that:

$$a_1 = \frac{5G}{A^2+B^2}$$

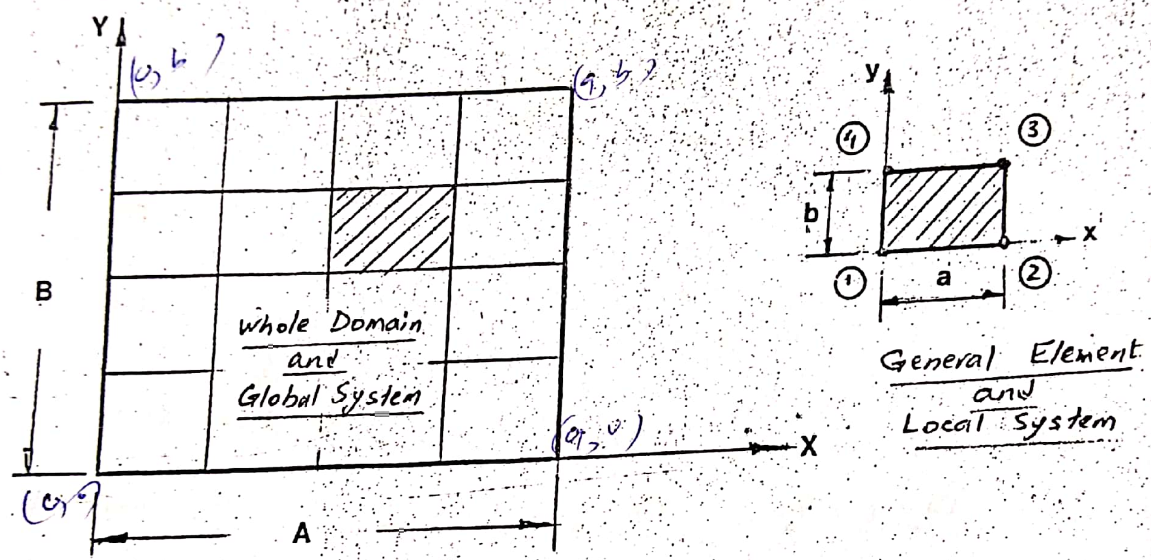
which is the same answer as obtained by Rayleigh-Ritz method.

2.5 FINITE ELEMENT SOLUTION

28.10.86

2.5.1 Formulation of Element Equations

Consider the same rectangular cross-section as the one solved by trial function methods. The domain can be discretised in terms of rectangular subdomains. A simple element, employed for such discretisation, is the 4-node rectangular element. A useful technique is to define the element in terms of local axes and a local numbering system, as shown below.



The equations of the 4-node element can be derived by applying any one of the variational or weighted-residual methods. When employing the discretised Rayleigh-Ritz method the following steps lead to the element equations. Some useful techniques will be explained during the derivation.

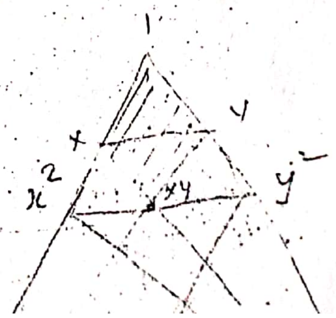
Step 1

Assume a solution in terms of nodal parameters and shape functions.

In order to illustrate the derivation of shape functions, the following trial solution can be assumed.

$$\omega(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$\omega(x,y) = \omega_1 N_1 + \omega_2 N_2 + \omega_3 N_3 + \omega_4 N_4$$



Using local system and nodal values, the following can be deduced

i) At $(x = 0, y = 0)$ $\omega = \omega_1$

hence $\omega_1 = \alpha_1$

$\boxed{x = a}$

$\omega_2 = \alpha_1 + \alpha_2 x$
$= \alpha_1 + \alpha_2 a$

ii) At $(x = a, y = 0)$ $\omega = \omega_2$

hence $\omega_2 = \alpha_1 + \alpha_2 a$

i.e. $\alpha_2 = (\omega_2 - \omega_1)/a$

$\omega = \underline{\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy}$

iii) At $(x = 0, y = b)$ $\omega = \omega_4$

hence $\omega_4 = \alpha_1 + \alpha_3 b$

i.e. $\alpha_3 = (\omega_4 - \omega_1)/b$

iv) At $(x = a, y = b)$ $\omega = \omega_3$

hence $\omega_3 = \alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 ab$

i.e. $\alpha_4 = (\omega_1 - \omega_2 + \omega_3 - \omega_4)/ab$

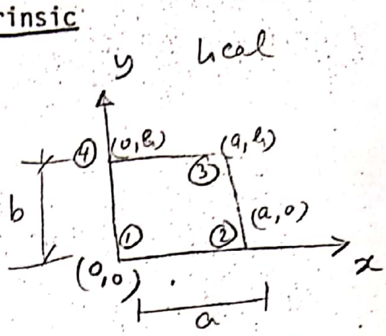
Finally, it can be deduced that

$$\omega\,(x,y) = \omega_1 + (\omega_2 - \omega_1)\left(\frac{x}{a}\right) + (\omega_4 - \omega_1)\left(\frac{y}{b}\right) + (\omega_1 - \omega_2 + \omega_3 - \omega_4)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$$

or

$$\begin{aligned}\omega\,(x,y) = {} & \left(1 - \frac{x}{a} - \frac{y}{b} + \frac{x}{a}\,\frac{y}{b}\right)\omega_1 \\ & + \left(\frac{x}{a} - \frac{x}{a}\,\frac{y}{b}\right)\omega_2 \\ & + \left(\frac{x}{a}\,\frac{y}{b}\right)\omega_3 \\ & + \left(\frac{y}{b} - \frac{x}{a}\,\frac{y}{b}\right)\omega_4\end{aligned}$$

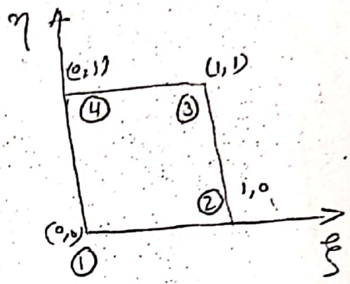
In order to simplify the expressions, the following intrinsic coordinates are defined



$$\xi = \frac{x}{a} \qquad \eta = \frac{y}{b}$$

Hence, it can be deduced that

$$\omega(\xi, \eta) = \sum_{i=1}^{4} \omega_i \; N_i(\xi, \eta)$$



a unique element

where

$$N_1(\xi, \eta) = (1 - \xi)(1 - \eta)$$

$$N_2(\xi, \eta) = \xi \; (1 - \eta)$$

$$N_3(\xi, \eta) = \xi \eta$$

$$N_4(\xi, \eta) = (1 - \xi) \; \eta$$

The previous expression can be written in matrix form as follows

$$\omega(\xi, \eta) = \underline{N}^t \underline{\omega} = \underline{\omega}^t \underline{N}$$

$\underline{w(\xi,\eta)} = w_1 N_1(\xi,\eta) + w_2 N_2(\xi,\eta) + w_3 N_3(\xi,\eta) + w_4 N_4(\xi,\eta)$

where

$\underline{\omega} = \{ \omega_1 \;\; \omega_2 \;\; \omega_3 \;\; \omega_4 \}$ ✓ column vector (independent) nodal value

$\underline{N} = \{ N_1 \;\; N_2 \;\; N_3 \;\; N_4 \}$ column vector (Shape function)

Step 2

Boundary condition verification.

yes

This should be postponed until the equations for the whole domain are assembled.

Step 3

Substitute the assumed solution in the variational statement.

The variational statement can be expressed as follows

$$\chi = \chi_1 + \chi_2$$

where

$$\chi_1 = \frac{1}{2G} \int_0^b \int_0^a \left(\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2 \right) dxdy$$

$$\chi_2 = -2 \int_0^b \int_0^a \omega \; dxdy$$

Let us try to express every component of the above two terms in a matrix form.

$N = N(x,y)$

Using $\overset{\text{scalar}}{\omega}(x,y) = [\, N_1 \;\; N_2 \;\; N_3 \;\; N_4 \,] \; \underline{\omega}$?

then by partial differentiation, it can be shown that,

$$\frac{\partial \omega}{\partial x} = \left(\frac{\partial N_1}{\partial x} \;\; \frac{\partial N_2}{\partial x} \;\; \frac{\partial N_3}{\partial x} \;\; \frac{\partial N_4}{\partial x} \right) \underline{\omega}$$ Constant.

$$\frac{\partial \omega}{\partial y} = \left(\frac{\partial N_1}{\partial y} \;\; \frac{\partial N_2}{\partial y} \;\; \frac{\partial N_3}{\partial y} \;\; \frac{\partial N_4}{\partial y} \right) \underline{\omega}$$

Hence, it can be deduced that

$$\begin{bmatrix} \dfrac{\partial \omega}{\partial x} \\[2ex] \dfrac{\partial \omega}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial x} \\[2ex] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_4}{\partial y} \end{bmatrix} \underline{\omega}$$

Defining the gradient vector $\underline{g}$, such that

$$\underline{g} = \{ \frac{\partial \omega}{\partial x} \quad \frac{\partial \omega}{\partial y} \}$$

it can be shown that

$$\boxed{\underset{2X1}{\underline{g}} = \underset{2X4}{\underline{B}} \quad \underset{4X1}{\underline{\omega}}}$$

vector of nodal values.

where

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

derivative of the shape functions.

The integrand of χ_1 can be expressed as follows

$$\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2 = \underline{g}^t \; \underline{g}$$

$$= (\underline{B} \; \underline{\omega})^t \; (\underline{B} \; \underline{\omega})$$

$$= \underline{\omega}^t \; \underline{B}^t \; \underline{B} \; \underline{\omega}$$

Hence,

$$\chi_1 = \frac{1}{2G} \int_0^b \int_0^a (\underline{\omega}^t \; \underline{B}^t \; \underline{B} \; \underline{\omega}) \, dxdy$$

since $\underline{\omega}$ is independent of x and y, the above equation can be rearranged as follows

$$\chi_1 = \tfrac{1}{2} \, \underline{\omega}^t \; \left(\int_0^b \int_0^a \frac{1}{G} \; \underline{B}^t \; \underline{B} \; dxdy \right) \underline{\omega}$$

order of multiplication should not be disturbed

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It is useful, from the programming point-of-view to express 1/G in matrix form.

Since,

$$\underline{B}^t_{4\times 2} \; \underline{B}_{2\times 4} = \underline{B}^t_{4\times 2} \; \underline{I}_{2\times 2} \; \underline{B}_{2\times 4}$$

where

$\underline{I}_{2\times 2}$ is the unit matrix of order 2, or

$$\underline{I}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also, since 1/G is a scalar quantity which can be multiplied by any one of the integrand matrices, it can be deduced that

$$\frac{1}{G} \; \underline{B}^t \; \underline{B} = \underline{B}^t \left(\frac{1}{G} \; \underline{I} \right) \underline{B}$$

$$= \underline{B}^t \; \underline{D} \; \underline{B}$$

where

$$\underset{2\times 2}{\underline{D}} = \frac{1}{G} \; \underline{I} = \begin{bmatrix} \frac{1}{G} & 0 \\ 0 & \frac{1}{G} \end{bmatrix} \checkmark$$

Hence,

$$X_1 = \tfrac{1}{2} \; \underline{\omega}^t \left(\int_0^b \int_0^a \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy \right) \underline{\omega}$$

For the second term,

$w(x,y)$ $\quad w(x,y) = \underline{N}^t \underline{\omega} = \underline{\omega}^t \underline{N}.$

$$X_2 = -2 \int_0^b \int_0^a \omega \; dxdy$$

$$= -2 \int_0^b \int_0^a \underline{\omega}^t \; \underline{N} \; dxdy$$

$$= -\underline{\omega}^t \left(\int_0^b \int_0^a 2\,\underline{N} \; dxdy \right)$$

Finally,

$$X = X_1 + X_2$$

$\underline{Q}$ mat.

$$= \tfrac{1}{2}\, \underline{\omega}^t \boxed{\left(\int_0^b \int_0^a \underline{B}^t \, \underline{D} \, \underline{B} \; dxdy \right)} \underline{\omega}$$

$$- \; \underline{\omega}^t \boxed{\left(\int_0^b \int_0^a 2\,\underline{N} \; dxdy \right)} = \underline{P} \text{ mat}$$

Step 4

Minimise X.

The minimisation conditions are

$$\frac{\partial X}{\partial \omega_1} = 0, \quad \frac{\partial X}{\partial \omega_2} = 0, \quad \frac{\partial X}{\partial \omega_3} = 0, \quad \frac{\partial X}{\partial \omega_4} = 0$$

Defining the notation

$$\frac{\partial X}{\partial \underline{\omega}} = \left\{ \frac{\partial X}{\partial \omega_1} \;\; \frac{\partial X}{\partial \omega_2} \;\; \frac{\partial X}{\partial \omega_3} \;\; \frac{\partial X}{\partial \omega_4} \right\}$$

$$\left\{\frac{\partial x}{\partial w}\right\}_{4\times 1} = \begin{bmatrix} \frac{\partial x}{\partial w_1} \\ \frac{\partial x}{\partial w_2} \\ \frac{\partial x}{\partial w_3} \\ \frac{\partial x}{\partial w_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

vector

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the minimisation conditions can be expressed in a matrix form as follows.

$$\frac{\partial X}{\partial \underline{\omega}} = \underline{0}$$

Zero matrix (4×1).

where

$$\underline{0} = \{0 \quad 0 \quad 0 \quad 0\}$$

Theorems

i) $\frac{\partial}{\partial \underline{\omega}} \left(\frac{1}{2} \underline{\omega}^t \underline{Q} \underline{\omega} \right) = \underline{Q} \underline{\omega}$

ii) $\frac{\partial}{\partial \underline{\omega}} \left(\underline{\omega}^t \underline{P} \right) = \underline{P}$

where $\underline{Q}$ and $\underline{P}$ are matrices which are independent of $\underline{\omega}$

Applying these theorems to the above expression of X, it can be deduced that

$$\frac{\partial X}{\partial \underline{\omega}} = \left(\int_0^b \int_0^a \underline{B}^t \underline{D} \underline{B} \, dxdy \right) \underline{\omega} - \int_0^b \int_0^a 2 \underline{N} \, dxdy = \underline{0}$$

i.e. $$\left(\int_0^b \int_0^a \underline{B}^t \underline{D} \underline{B} \, dxdy \right) \underline{\omega} = \int_0^b \int_0^a 2 \underline{N} \, dxdy$$

or $$\underline{K}_{(e)} \underline{\omega} = \underline{F}$$

where

$$\underline{K}_{(e)} = \int_0^b \int_0^a \underline{B}^t \underline{D} \underline{B} \, dxdy$$

which is the element stiffness matrix.

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$$\underline{F}_{(e)} = \int_0^b \int_0^a 2\,\underline{N}\ dxdy$$

which is the element nodal force vector.

The previous expression for the element stiffness matrix is, more or less, similar to that employed for many other applications. A computer program is required for the generation of the $\underline{B}$ and $\underline{D}$ matrices and to perform the double integration.

Explicit Form

An explicit form, for hand calculations, can be deduced as follows:

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

$$\underline{B}^t = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} \\ \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

Hence,

$$\underline{K}_{(e)} = \frac{1}{G}\int_0^b\int_0^a \underline{B}^t\ \underline{B}\ dxdy$$

$$= \frac{1}{G}\int_0^b\int_0^a \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} \\ \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix} dxdy$$

It can be deduced that

$$K_{ij} = \frac{1}{G}\int_0^b\int_0^a \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

Similarly, it can be shown that

$$F_i = \int_0^b\int_0^a 2\ N_i\ dxdy$$

<u>Use of ξ η</u>

From- the definition of ξ and η

$$\xi = x/a, \qquad x = a\xi$$

$$\eta = y/b, \qquad y = b\eta$$

$dx = a\,d\xi$

$dy = b\,d\eta$.

Also,

$$F_2 = 2ab \int_0^1 \int_0^1 \xi(1-\eta)\, d\xi d\eta$$

$$= \frac{ab}{2}$$

Exercise 2.3

Prove that the stiffness matrix of the 4 node rectangular element is

$$\underline{K}_{(e)} = \frac{1}{6G} \begin{bmatrix} 2\alpha+2\beta & \alpha-2\beta & -\alpha-\beta & -2\alpha+\beta \\ \alpha-2\beta & 2\alpha+2\beta & -2\alpha+\beta & -\alpha-\beta \\ -\alpha-\beta & -2\alpha+\beta & 2\alpha+2\beta & \alpha-2\beta \\ -2\alpha+\beta & -\alpha-\beta & \alpha-2\beta & 2\alpha+2\beta \end{bmatrix}$$

and the nodal force vector is

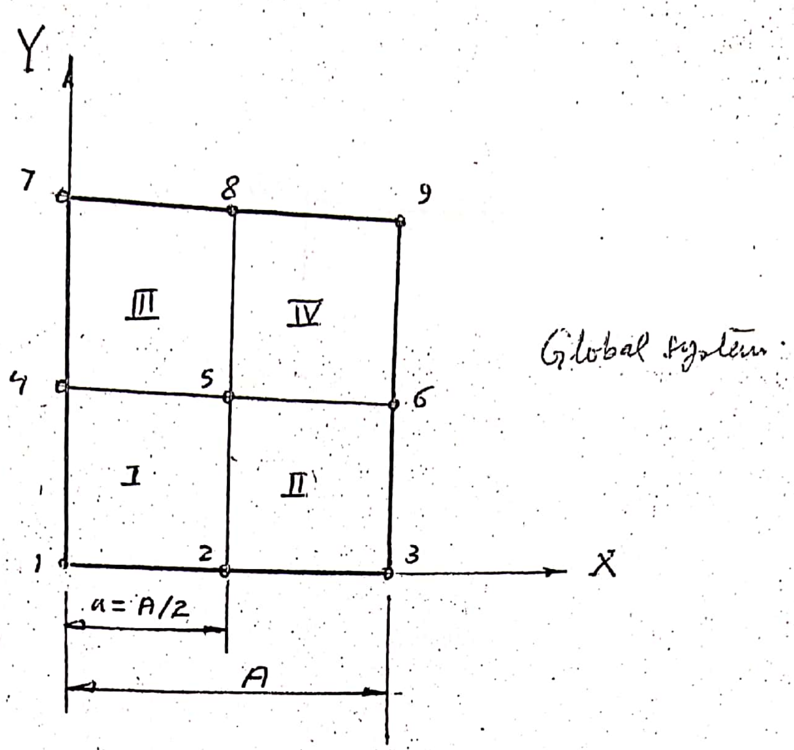
$$\underline{F}_{(e)} = \frac{ab}{2} \{ 1 \quad 1 \quad 1 \quad 1 \}$$

2.5.2 Steps of the Solution for the Whole Domain

Step 1

Discretisation of the domain.

Consider a square cross-section divided into 4 similar square element, as shown below:



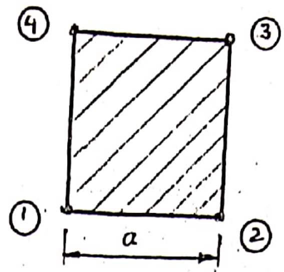
The discretised domain is known as the finite element mesh and the discretisation process is called mesh generation.

Step 2

Formulation of element equations.

For any 4-node square element, it can be shown that

$$\underline{K}^{(e)} = \frac{1}{6G}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$



$$F_{(e)} = \frac{a^2}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that the element nodal parameter vector should follow the local order of the element nodes. Hence, it can be shown that

a) <u>for element 1</u>

$$\underline{K}_{(I)} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_5 \\ \omega_4 \end{bmatrix} = \underline{F}_{(I)}$$

b) <u>for element II</u>

$$\underline{K}_{(II)} \begin{bmatrix} \omega_2 \\ \omega_3 \\ \omega_6 \\ \omega_5 \end{bmatrix} = \underline{F}_{(II)}$$

etc.

Step 3

Assembly of the equations for the whole domain.

A unique nodal vector

the load vector is composed of the external nodal forces as specified in the input file of the assembly of the element loads

$$\underline{\omega} = \{ \omega_1 \quad \omega_2 \quad \dots \quad \omega_9 \}$$

The nodal load vector (shear force and bending moment) are obtained from the shape functions of 2-node Beam element

should be used for each element. The element matrix $\underline{K}_{(e)}$ should be expanded to be 9 x 9, by inserting the correct number of zeros. Then,

$$\underset{\text{whole domain}}{\underline{K}} = \sum_{e=1}^{4} K_{(e)}$$

$$\underset{\text{whole domain}}{\underline{F}} = \sum_{e=1}^{4} \underline{F}_{(e)}$$

A tabulated form can be used as shown in the following figure. For real problems, the operations should be carried out by means of a digital computer.

The assembled equations are

$$\frac{1}{6G}\left[\begin{array}{ccc|ccc|ccc} +4 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & -2 & -2 & -2 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & -2 & -1 & 0 & 0 & 0 \\ \hline -1 & -2 & 0 & 8 & -2 & 0 & -1 & -2 & 0 \\ -2 & -2 & -2 & -2 & 16 & -2 & -2 & -2 & -2 \\ 0 & -2 & -1 & 0 & -2 & 8 & 0 & -2 & -1 \\ \hline 0 & 0 & 0 & -1 & -2 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & 4 \end{array}\right] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \boxed{\omega_5} \\ \omega_6 \\ \omega_7 \\ \omega_8 \\ \omega_9 \end{bmatrix} = \frac{a^2}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 4 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\frac{1}{6G}(16)\,\omega_5 = \frac{a^2}{2}(4)$$

$$\omega_5 = 2a^2 \cdot \frac{3G}{8} = \left(\frac{3a^2}{4}\right)G$$

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ASSEMBLY TABLE

$\times \frac{1}{6G}$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	F
F_1	4	-1		-1	-2					1
F_2	$\boxed{-1}$	4 + 4	$\boxed{-1}$	-2	-1 $\boxed{-1}$	-2				1 +1
F_3		$\cancel{1}$ $\boxed{-1}$	4		-2	-1				$\boxed{1}$
F_4	-1	-2		4 $\boxed{+4}$	-1 $\boxed{-1}$		$\boxed{-1}$	$\boxed{-2}$		1 $\boxed{+1}$
F_5	-2	-1 -1	-2	-1 $\boxed{-1}$	4 +4 $\boxed{+4}$ $\boxed{+4}$	-1 $\boxed{-1}$	$\boxed{-2}$	$\boxed{-1}$ $\boxed{-1}$	$\boxed{-2}$	1 +1 $\boxed{+1}$ $\boxed{+1}$
F_6		-2	-1		-1 $\boxed{-1}$	4 $\boxed{+4}$		$\boxed{-2}$	$\boxed{-1}$	$\boxed{1}$ $\boxed{+1}$
F_7				$\boxed{-1}$	$\boxed{-2}$		$\boxed{4}$	$\boxed{-1}$		$\boxed{1}$
F_8				$\boxed{-2}$	$\boxed{-1}$ $\boxed{-1}$	$\boxed{-2}$	$\boxed{-1}$	$\boxed{4}$ $\boxed{+4}$	$\boxed{-1}$	$\boxed{1}$ $\boxed{+1}$
F_9					$\boxed{-2}$	$\boxed{-1}$		$\boxed{-1}$	$\boxed{4}$	$\boxed{1}$

$\frac{a^2}{2}$

4

Step 4

Apply the boundary conditions.

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_6 = \omega_7 = \omega_8 = \omega_9 = 0$$

The remaining equation is

i.e. $$\boxed{K_{55} \ \omega_5 = F_5}$$

$$16 \ \omega_5 = 4(3 \ Ga^2)$$

$\boxed{F_i = \frac{A}{2}}$

Step 5

Solve the resulting reduced equation(s).

$$\omega_5 = \frac{3.G \ a^2}{4} = \frac{3G \ A^2}{16}$$

$$T = \Sigma \ (\iint 2 \ \omega \ dA) \ \theta$$

For any element

$$\iint 2 \ \omega dA = \sum_{i=1}^{4} (\omega_i)_e \ (F_i)_e$$

i.e.

$$\Sigma \iint 2 \ \omega dA = \sum_{i=1}^{9} \omega_i \ F_i = \omega_5 \ F_5$$

$$= \frac{3G \ A^2}{16} \cdot \frac{A^2}{2}$$

$$T = \frac{3G \ A^4}{32} \cdot \theta$$

Constant

2.6 EXAMPLE OF STEADY-STATE HEAT CONDUCTION

2.6.1 Formulation of Element Equations

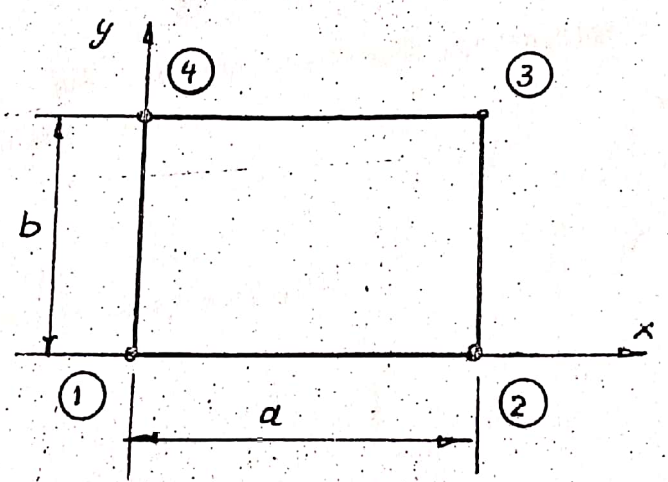
Steady-state heat conduction problems are governed by the following equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right) + Q = 0$$

or

$$X = \tfrac{1}{2} \iint_{\text{Domain}} \left[k_x \left(\frac{\partial T}{\partial x}\right)^2 + k_y \left(\frac{\partial T}{\partial y}\right)^2\right] dxdy$$

$$- \iint_{\text{Domain}} Q\ T\ dxdy \; = \; \text{extremum}$$



Local System

Using the steps of the discretisied Rayleigh-Ritz method on the 4-node rectangular element.

Step 1

Assume a solution in terms of nodal parameters and shape functions.

Q These are shape functions of 4 node rectangular element

$$T(x,y) = T_1 N_1(x,y) + T_2 N_2(x,y) + T_3 N_3(x,y) + T_4 N_4(x,y)$$

As for the previous case, it can be shown that

$$N_1 = (1 - \frac{x}{a})(1 - \frac{y}{b}) = (1 - \xi)(1 - \eta)$$

$$N_2 = \frac{x}{a}(1 - \frac{y}{b}) = \xi(1 - \eta)$$

$$N_3 = \frac{x}{a}\frac{y}{b} = \xi\eta$$

$$N_4 = (1 - \frac{x}{a})\frac{y}{b} = (1 - \xi)\eta$$

where

$$\xi = x/a, \qquad \eta = y/b$$

Hence, for all field problems the 4-node rectangular element has the same shape functions.

<u>Step 2</u>

Boundary condition verification.

(Postponed)

<u>Step 3</u>

Substitute the assumed solution in the variational statement.

$$\chi = \chi_1 + \chi_2$$

where

$$\chi_1 = \frac{1}{2}\int_0^b\int_0^a \left[k_x \left(\frac{\partial T}{\partial x}\right)^2 + k_y \left(\frac{\partial T}{\partial y}\right)^2 \right] dxdy$$

$$\chi_2 = - \int_0^b \int_0^a Q\, T \, dxdy$$

The integrand of χ_1

$$= k_x \left(\frac{\partial T}{\partial x}\right)^2 + k_y \left(\frac{\partial T}{\partial y}\right)^2$$

$$= \begin{bmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} \end{bmatrix} \begin{bmatrix} k_x \frac{\partial T}{\partial x} \\ k_y \frac{\partial T}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

$$= \underline{g}^t \, \underline{D} \, \underline{g}$$

where

$$\underline{g} = \left\{ \frac{\partial T}{\partial x} \quad \frac{\partial T}{\partial y} \right\}$$

$$\underline{D} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

From the previous torsional problem, it can be deduced that:

$$\underline{g} = \underline{B} \, \underline{T}$$

where

$$\underline{T} = \{ T_1 \quad T_2 \quad T_3 \quad T_4 \}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}$$

Hence,

$$\underline{g}^t \; \underline{D} \; \underline{g} = \underline{T}^t \; \underline{B}^t \; \underline{D} \; \underline{B} \; \underline{T}$$

and

$$\chi_1 = \tfrac{1}{2} \; \underline{T}^t \left(\int_0^b \int_0^a \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy \right) \underline{T}$$

Similarly,

$$\chi_2 = -\underline{T}^t \left(\int_0^b \int_0^a Q \; \underline{N} \; dxdy \right)$$

and

$$\chi = \chi_1 + \chi_2$$

$$= \tfrac{1}{2} \; \underline{T}^t \left(\int_0^b \int_0^a \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy \right) \underline{T}$$

$$- \underline{T}^t \left(\int_0^b \int_0^a Q \; \underline{N} \; dxdy \right)$$

Step 4

Extremise χ

$$\frac{\partial \chi}{\partial \underline{T}} = \underline{0}$$

Using the previous theorems, it can be deduced that:

$$\frac{\partial X}{\partial \underline{T}} = \left(\int_0^b \int_0^a \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy \right) \underline{T} - \int_0^b \int_0^a Q \underline{N} \; dxdy = \underline{0}$$

i.e.

$$\underline{K}_{(e)} \; \underline{T}' = \underline{F}_{(e)}$$

where

$$\underline{K}_{(e)} = \int_0^b \int_0^a \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy$$

$$\underline{F}_{(e)} = \int_0^b \int_0^a Q \; \underline{N} \; dxdy$$

For the special case of $k_x = k_y = k$, it can be proved that

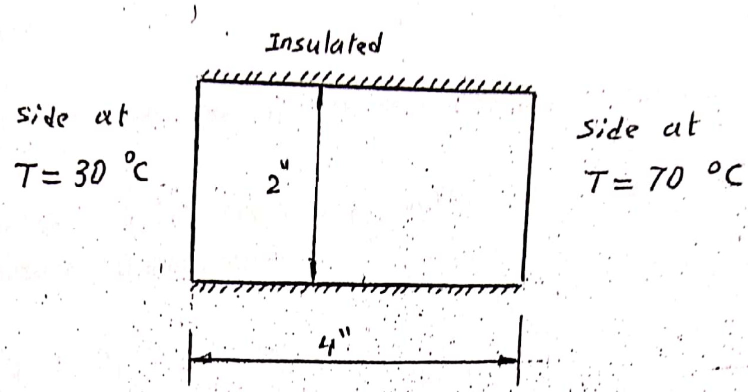
$$\underline{K}_{(e)} = \frac{k}{6} \begin{bmatrix} 2\alpha+2\beta & \alpha-2\beta & -\alpha-\beta & -2\alpha+\beta \\ \alpha-2\beta & 2\alpha+2\beta & -2\alpha+\beta & -\alpha-\beta \\ -\alpha-\beta & -2\alpha+\beta & 2\alpha+2\beta & \alpha-2\beta \\ -2\alpha+\beta & -\alpha-\beta & \alpha-2\beta & 2\alpha+2\beta \end{bmatrix}$$

where

$$\alpha = a/b$$
$$\beta = b/a$$

2.6.2 Finite Element Solution

The element equations previously formulated are now used to find the temperature distribution for the given domain. Consider heat conduction in the domain shown below



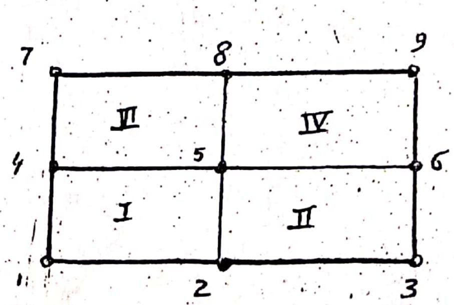
For such a problem, the heat generated is unknown at any node where the temperature is specified.

Applying the standard steps of the FEM solution.

Step 1

Discretisation of the domain.

The domain is to be divided into a suitable number of finite elements connected by nodes. The more elements employed, the more accurate will be the temperature distribution. For the sake of hand calculations, let us consider the following simple finite element mesh, which consists of four rectangular elements connected with 9 nodes.



<u>Step 2</u>

Formulation of Equations for each finite element in the mesh.

From the results of 2.6.1, the matrix equation for a general 4-node rectangular element with

$$a = 2, \qquad b = 1$$

is as follows

$$\underline{K}_{(e)} \quad \underline{T}_{(e)} = \underline{F}_{(e)}$$

where

$$\underline{K}_{(e)} = \frac{k}{6}\begin{bmatrix} 5 & 1 & -2.5 & -3.5 \\ 1 & 5 & -3.5 & -2.5 \\ -2.5 & -3.5 & 5 & 1 \\ -3.5 & -2.5 & 1 & 5 \end{bmatrix}$$

Applying the above matrix formulation for the given elements in the mesh, the following can be deduced:

a) <u>Element I</u>

$$\underline{K}_{(e)} \begin{bmatrix} T_1 \\ T_2 \\ T_5 \\ T_4 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_5 \\ Q_4 \end{bmatrix}_{I}$$

b) <u>Element II</u>

$$\underline{K}_{(e)} \begin{bmatrix} T_2 \\ T_3 \\ T_6 \\ T_5 \end{bmatrix} = \begin{bmatrix} Q_2 \\ Q_3 \\ Q_6 \\ Q_5 \end{bmatrix}_{II}$$

c) Element III

$$\underline{K}_{(e)} \begin{bmatrix} T_4 \\ T_5 \\ T_8 \\ T_7 \end{bmatrix} = \begin{bmatrix} Q_4 \\ Q_5 \\ Q_8 \\ Q_7 \end{bmatrix}_{III}$$

d) Element IV

$$\underline{K}_{(e)} \begin{bmatrix} T_5 \\ T_6 \\ T_9 \\ T_8 \end{bmatrix} = \begin{bmatrix} Q_5 \\ Q_6 \\ Q_9 \\ Q_8 \end{bmatrix}_{IV}$$

Step 3

Assembly of the equations for the whole domain.

From the given table, the equations for the whole domain are:

ASSEMBLY TABLE

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
Q_1	5	1		-3.5	-2.5				
Q_2	1	5 5	1	-2.5	-3.5 -3.5	-2.5			
Q_3		1	5		-2.5	-3.5			
Q_4	-3.5	-2.5		5+5	1+1		-3.5	-2.5	
Q_5	-2.5	-3.5 -3.5	-2.5	1+1	5+5 5+5	1+1	-2.5	-3.5 -3.5	-2.5
Q_6		-2.5	-3.5		1+1	5+5		-2.5	-3.5
Q_7				-3.5	-2.5		5	1	1
Q_8				-2.5	-3.5 -3.5	-2.5	1	5 +5	1
Q_9					-2.5	-3.5		1	5

$$\frac{k}{6}\begin{bmatrix} 5 & 1 & 0 & -3.5 & -2.5 & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & -2.5 & -7 & -2.5 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & -2.5 & -3.5 & 0 & 0 & 0 \\ -3.5 & -2.5 & 0 & 10 & 2 & 0 & -3.5 & -2.5 & 0 \\ -2.5 & -7 & -2.5 & 2 & 20 & 2 & -2.5 & -7 & -2.5 \\ 0 & -2.5 & -3.5 & 0 & 2 & 10 & 0 & -2.5 & -3.5 \\ 0 & 0 & 0 & -3.5 & -2.5 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & -2.5 & -7 & -2.5 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & -2.5 & -3.5 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix}$$

$$= \{ Q_1 \quad Q_2 \quad .. \quad Q_9 \}$$

Step 4

Boundary condition verification.

$$T_1 = T_4 = T_7 = 30, \qquad Q_1, Q_4, Q_7 = \text{Unknown}$$

$$T_3 = T_6 = T_9 = 70, \qquad Q_3, Q_6, Q_9 = \text{Unknown}$$

$$(Q_2 = Q_5 = Q_8 = 0)$$

Eliminating equations 1, 4, 7, 3, 6, 9 the reduced system can be expressed as follows

$$\begin{bmatrix} 10 & -7 & 0 \\ -7 & 20 & -7 \\ 0 & -7 & 10 \end{bmatrix} \begin{bmatrix} T_2 \\ T_5 \\ T_8 \end{bmatrix} = \begin{bmatrix} 150 \\ 300 \\ 150 \end{bmatrix}$$

Step 5

Solution of the resulting equations.

The resulting reduced equations are

$$10\,T_2 - 7\,T_5 = 150$$

$$-7\,T_2 + 20\,T_5 - 7\,T_8 = 300$$

$$-7\,T_5 + 10\,T_8 = 150$$

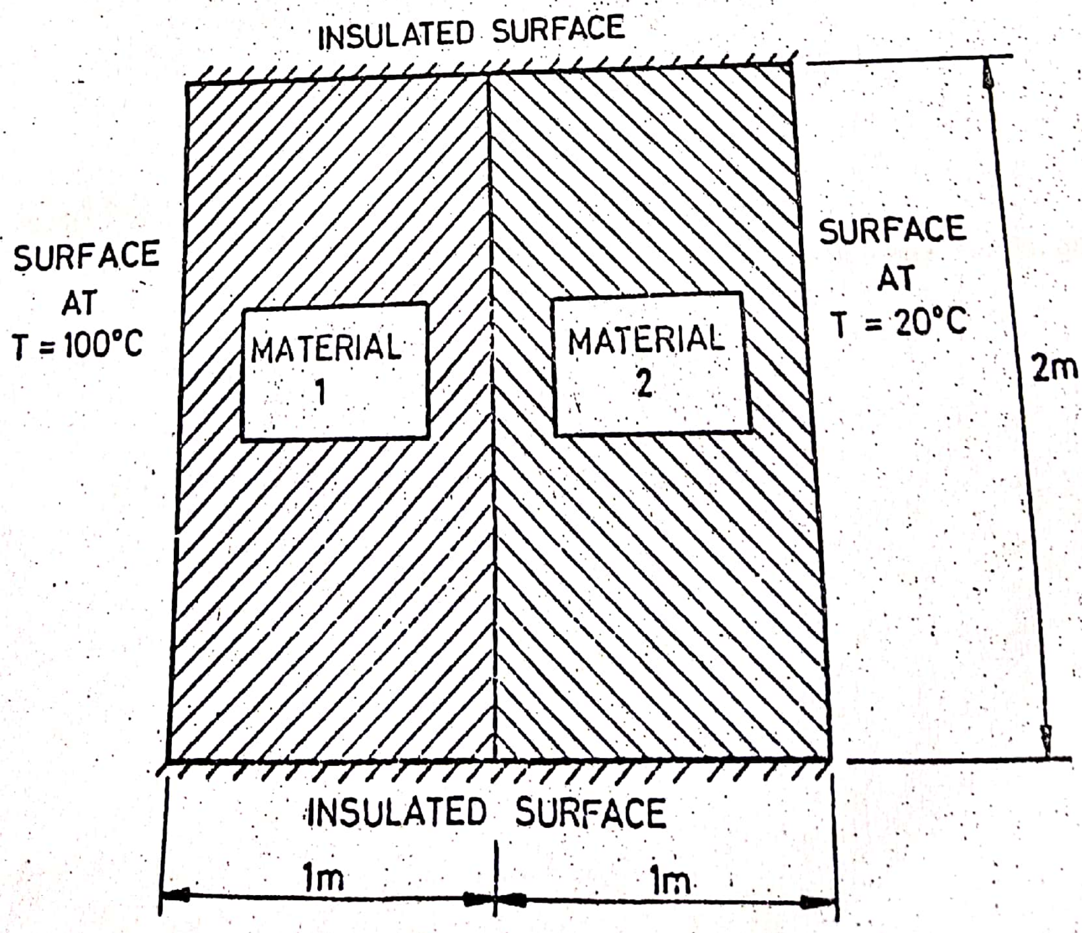
which have the following solution

$$T_2 = T_5 = T_8 = 50^{o}C$$

Exercise 2.5 1984 Important

The Finite Element Method is a general numerical technique which is capable of solving any boundary value problem.

To verify the above, it is required to solve the following steady-state heat conduction problem in the two dimensional wall, shown below, using the Finite Element Method.



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Material 1

$$k_x = k_y = 24 \text{ Watt/m/}^oC$$

Material 2

$$k_x = k_y = 36 \text{ Watt/m/}^oC$$

Governing Equations

For steady-state, two-dimensional, heat conduction, the differential equation is

$$\frac{\partial}{\partial x}(K_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial T}{\partial y}) + Q = 0$$

where

k_x, k_y are the heat conductivities in the x and y directions,

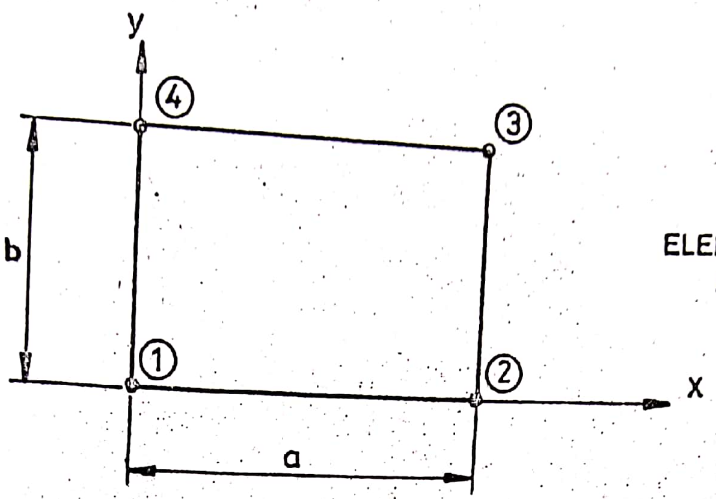
Q is the heat generated per unit volume.

The equivalent variational statement can be expressed as follows:

$$\chi = \frac{1}{2} \iint_{\text{Domain}} [k_x (\frac{\partial T}{\partial x})^2 + k_y (\frac{\partial T}{\partial y})^2] dxdy - \iint_{\text{Domain}} Q\, T\, dxdy = \text{extremum.}$$

REQUIREMENTS

a) If the problem is to be solved by means of the 4-node rectangular element shown below:



ELEMENT LOCAL SYSTEM

i) Starting with

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

Prove that the field function $T(x,y)$ can be expressed in terms of nodal values and shape functions as follows

$$T(x,y) = T_1 \; N_1(x,y) + T_2 \; N_2(x,y) + T_3 \; N_3(x,y) + T_4 \; N_4(x,y)$$

where

$$N_1 = (x-a)\;(y-b)/(ab)$$

$$N_2 = -x\;(y-b)/(ab)$$

$$N_3 = xy/(ab)$$

$$N_4 = -(x-a)\;y/(ab)$$

ii) Show that the matrix equations of the element can be expressed as follows

$$\underline{K}_{(e)}\ \underline{T}_{(e)} = \underline{Q}_{(e)}$$

where

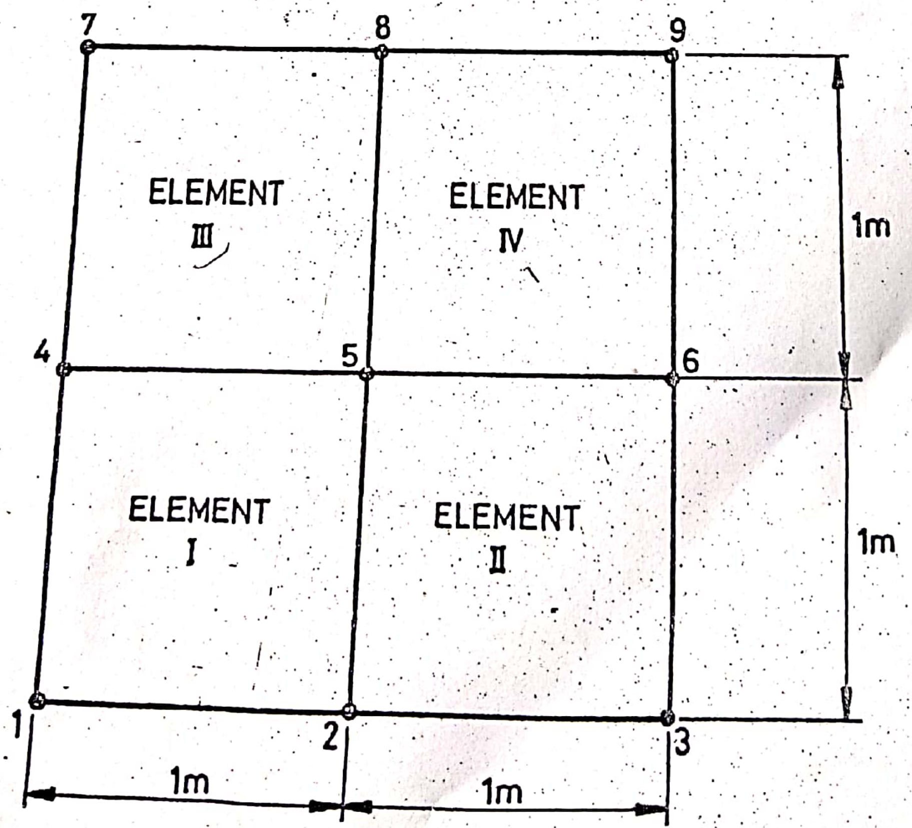
$$\underline{T}_{(e)} = \{T_1 \quad T_2 \quad T_3 \quad T_4\}$$

$$\underline{K}_{(e)} = \frac{k}{6}\begin{bmatrix} 2\alpha+2\beta & \alpha-2\beta & -\alpha-\beta & -2\alpha+\beta \\ \alpha-2\beta & 2\alpha+2\beta & -2\alpha+\beta & -\alpha-\beta \\ -\alpha-\beta & -2\alpha+\beta & 2\alpha+2\beta & \alpha-2\beta \\ -2\alpha+\beta & -\alpha-\beta & \alpha-2\beta & 2\alpha+2\beta \end{bmatrix}$$

$$k_x = k_y = k$$

$$\alpha = a/b$$
$$\beta = b/a$$

b) Using the mesh shown below (or any other mesh), find the temperature distribution on the common surface between the two materials of the wall.



$$T_1 = T_4 = T_7 = 100, \qquad T_3 = T_6 = T_9 = 20$$

3. ONE-DIMENSIONAL ELEMENTS

APPLICATION TO FRAMED STRUCTURES

3.1 LAGRANGIAN ELEMENTS

3.1.1 Introductory Case Study General Bar Element

The derivation of the 2-node bar element is discussed in Chapter 1. If the bar has a variable cross-sectional area, the accuracy of the finite element solution, for the same number of elements, can be improved by using an element with more than two nodes. An n-node bar element is used for demonstrating Lagrangian one-dimensional elements. The following standard algorithm is employed for the derivation of element equations.

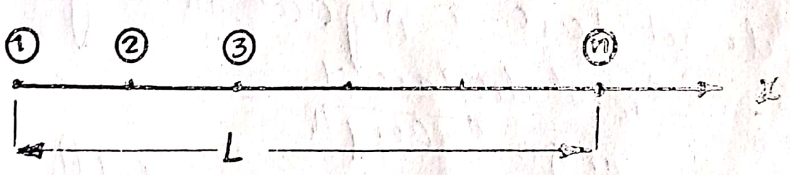
force vector

check

Step 1

Define the nodal parameters for the element.

For an n-node element, with a local numbering system as shown below, the following vectors are defined



$$\underline{\delta} = \{ u_1 \quad u_2 \quad \dots \quad u_n \}$$

$$\underline{F} = \{ F_1 \quad F_2 \quad \dots \quad F_n \}$$

where u_i is the displacement value in the x-direction, at the i^{th} node

F_i is the force value in the x-direction, at the i^{th} node.

The actual loading system should be replaced by an equivalent nodal system. The rules for such equivalence will be discussed in Chapter 5.

Step 2

Express the displacement at any point in terms of nodal displacements and shape functions.

The displacement field function can be assumed in terms of the following algebraic polynominal.

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots + \alpha_n x^{n-1}$$

or

$$u(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^{n-1} \end{bmatrix} \underline{\alpha}$$

where

$$\underline{\alpha} = \{ \alpha_1 \ \alpha_2 \ \ldots \ \alpha_n \}.$$

At $x = x_1$, $x = x_2$, ..., $x = x_n$, the following n conditions are obtained.

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_1^2 + \ldots + \alpha_n x_1^{n-1}$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 x_2^2 + \ldots + \alpha_n x_2^{n-1}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$u_n = \alpha_1 + \alpha_2 x_n + \alpha_3 x_n^2 + \ldots + \alpha_n x_n^{n-1}$$

which can be represented in the following matrix form

$$\underline{C}\ \underline{\alpha} = \underline{\delta}$$

where

$$C_{ij} = x_i^{j-1} \qquad \underline{\alpha} = \underline{C}^{-1} \underline{\delta}$$

$i = 1, 2, \ldots, n$

$j = 1, 2, \ldots, n$

The above matrix equation has the following solution

$$\underline{\alpha} = \underline{C}^{-1} \underline{\delta}$$

Hence,

$$u(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^{n-1} \end{bmatrix} \underline{C^{-1} \delta}$$

Defining the shape functions $N_1, N_2, \ldots, N_n$, such that

$$u(x) = \sum_{i=1}^{n} u_i N_i$$

$$= \underline{N}^t \underline{\delta} = \underline{\delta}^t \underline{N}$$

where

$$\underline{N} = \{ N_1 \quad N_2 \quad \ldots \quad N_n \}$$

it can be proved that

C^{-1}_{ji}

$$N_i(x) = \sum_{j=1}^{n} C^{-1}_{j,i} \, x^{j-1}$$

where $C^{-1}_{j,i}$ is the value at the j^{th} row and i^{th} column for the inverse matrix $\underline{C}^{-1}$

Step 3

Express the strain(s) at any point in terms of nodal displacements and shape functions.

From st

non-ze

From strain-displacement relationships, it can be shown that the non-zero strain component is

$$\varepsilon_x = \frac{du}{dx} = \sum_{i=1}^{n} u_i \frac{dN_i}{dx}$$

which can be expressed in matrix form as follows

$$\underline{\varepsilon} = \underline{B}\ \underline{\delta}$$

where

$$\underline{\varepsilon} = \{\ \varepsilon_x\ \}$$

$$\underline{B} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \cdots & \frac{dN_n}{dx} \end{bmatrix}$$

Step 4

Express the stress(es) at any point in terms of nodal displacements and shape functions

For a linear elastic material, it can be shown that

$$\sigma_x = E\ \varepsilon_x$$

where

E = Young's modulus for the material.

Hence $\underline{\sigma} = \underline{D}\ \underline{\varepsilon} = \underline{D}\ \underline{B}\ \underline{\delta}$

where

$$\underline{\sigma} = \{\ \sigma_x\ \}$$

$$\underline{D} = [E]$$

Step 5

Express the total potential energy of the element in terms of nodal displacements.

$$\chi = U - W$$

where

χ is the total potential energy of the element,
U is the element strain energy, and
W is the work done by external loads

For a linear elastic material, it can be shown that

$$U = \frac{1}{2} \iiint_{\text{element}} \underline{\varepsilon}^t \, \underline{\sigma} \, dxdydz$$

Hence,

$$U = \frac{1}{2} \underline{\delta}^t \left(\iiint_{\text{element}} \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta}$$

The definition of the equivalent loading vector $\underline{F}$ implies that

$$W = \underline{\delta}^t \, \underline{F}$$

Hence,

$$\chi = \frac{1}{2} \underline{\delta}^t \left(\iiint_{\text{element}} \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta} - \underline{\delta}^t \, \underline{F}$$

Step 6

Apply the minimum total potential energy theorem / Extras λ

$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{0}$$

Hence, it can be shown that

$$\left(\iiint_{\text{element}} \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta} - \underline{F} = \underline{0}$$

i.e.
$$\underline{K} \, \underline{\delta} = \underline{F}$$

where

$$\underline{K} = \iiint_{\text{element}} \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz$$

which is the element stiffness matrix

3.2 HERMITIAN SHAPE FUNCTIONS. BEAM-BENDING ELEMENT

a) Beam-Bending Theory

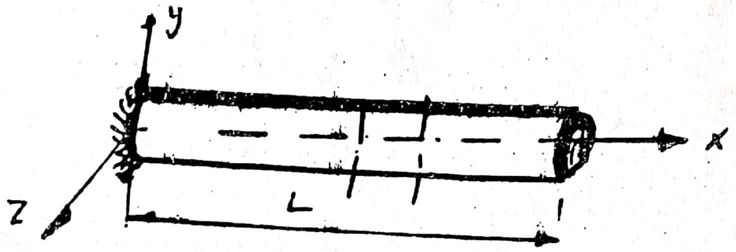
The conventional Euler-Bernouilli beam-bending theory is based upon the following assumptions

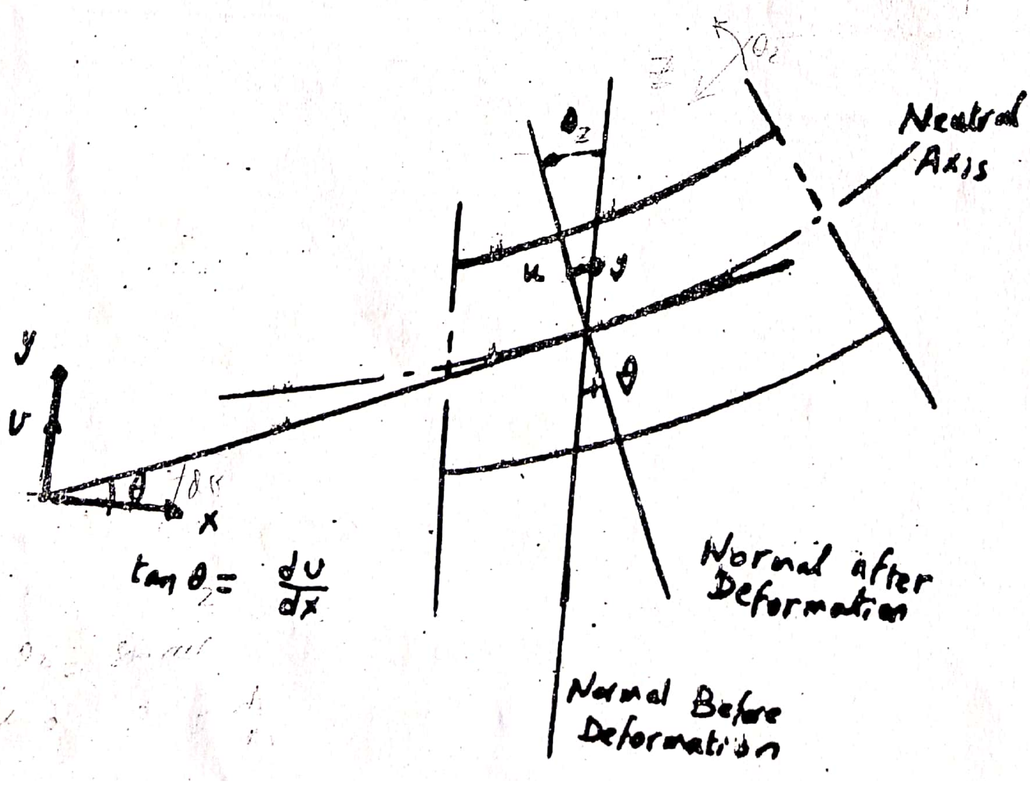
i) Lateral deflections are small compared with the beam thickness.

ii) The beam material is homogeneous, isotropic, and linearly elastic.

iii) Transverse shear is negligible.

iv) Plane cross-sections remain plane after loading.





$\tan\theta_z = \frac{dv}{dx}$

Consider the beam shown in the above figure. At any point (x, y, z) inside the beam where the x-axis is assumed to be the beam neutral-axis, the displacement components can be assumed as follows

i) The vertical displacement, along the y-axis, approximates to the vertical displacement of the neutral axis.

ii) The z-component is negligible.

iii) The x-component is a function of y and the rotation angle.

Hence, it can be deduced that

$\tan\theta = \frac{dv}{dx}$

$$u = -y \frac{dv}{dx}$$

$$v = v(x)$$

$$w = 0$$

$u(x,y) = -y \frac{dv}{dx}$

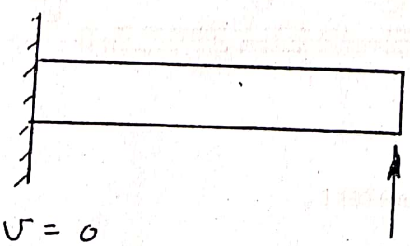
$v(x,y) = v(x)$

$w(x,y) = 0$

where u, v, and w are x, y and z displacement components, respectively.

It is clear that the field function of the beam-bending problem is the vertical displacement v.

Consider a cantilever as shown below



$v = 0$

$\frac{dv}{dx} = 0$

?

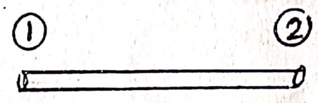
The boundary conditions are

$$v = 0, \qquad \frac{dv}{dx} = 0$$

at the fixed end. This case represents a class of problems where the boundary conditions are on the field function and its first order derivative. Since the basic advantage of pointwise discretisation is to obtain automatic satisfaction of the boundary conditions, the nodal parameters for such a class should contain both the field function and its first order derivative. If the domain is discretised piecewise into 2-node elements, the nodal parameters will be

$$\{ v_1 \quad \theta_1 \qquad v_2 \quad \theta_2 \}$$

where 1 and 2 are the local numbers of element nodes.



Vector

Note

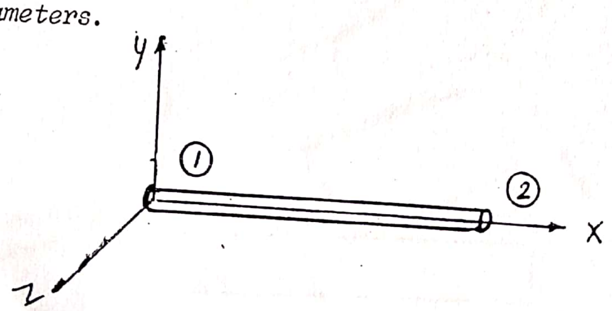
The matrix equations for such an element can be obtained by means of the discretised Rayleigh-Ritz method. However, it is useful to employ a standard algorithm which is applicable to most structural analysis problems.

2-node Beam element

b) Steps of Element Stiffness Matrix Derivation

Step 1

Define the nodal parameters.



Local System

There are two types of nodal parameters,

i) Nodal Displacements

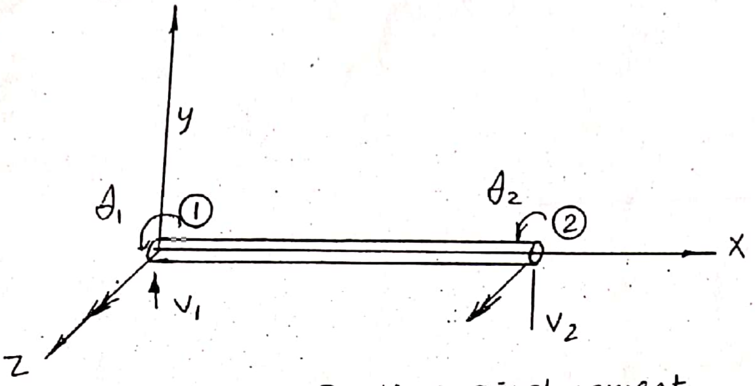
These are defined by the following vector:

$$\underline{\delta} = \{ \; v_1 \; \theta_1 \quad v_2 \; \theta_2 \; \}$$

Vectorial signs are employed as shown below

Note that

$$\theta = \frac{dv}{dx}$$



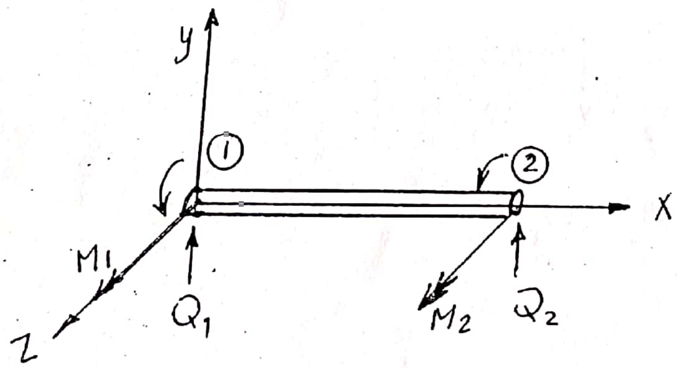
Postive Displacement.

ii) Nodal Forces

These are defined by the nodal loading vector,

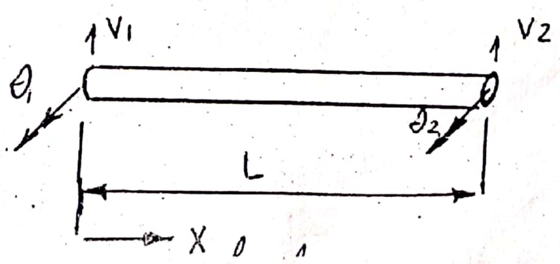
$$\underline{F} = \{ \; Q_1 \; M_1 \quad Q_2 \; M_2 \; \}$$

Vectorial signs are also used as shown in the figure below



Step 2

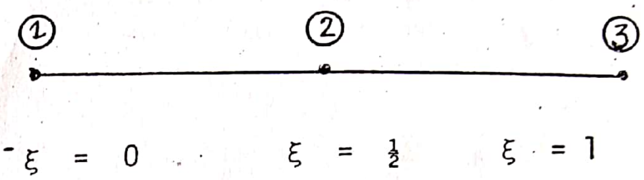
Express the displacement at any point in terms of nodal displacement and shape functions.



and

$$\mathcal{L}_j^n(x) = \prod_{\substack{R=1 \\ R\neq j}}^{n} \left(\frac{x-x_R}{x_j-x_R} \right)$$

Example - Three-Node Element



Using the Hermite's interpolation formula, for a 3-node element, the shape functions can be expressed in terms of ξ as follows

$$g_1(\xi) = (1 + 6\xi)\,(1 - \xi)^2\,(1 - 2\xi)^2$$

$$h_1(\xi) = \xi\,(1 - \xi)^2\,(1 - 2\xi)^2$$

$$g_2(\xi) = 16\,\xi^2\,(1 - \xi)^2$$

$$h_2(\xi) = -8\,\xi^2\,(1 - \xi)^2\,(1 - 2\xi)$$

$$g_3(\xi) = (7 - 6\xi)\,\xi^2\,(1 - 2\xi)^2$$

$$h_3(\xi) = -\,\xi^2\,(1 - \xi)\,(1 - 2\xi)^2$$

Step 3

Express the strains at any point in terms of nodal displacements.

From strain displacement relationships

$$\varepsilon_x = \frac{\partial u}{\partial x} = -y\frac{d^2v}{dx^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{dv}{dx} + \frac{dv}{dx} = 0$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

The vector of relevant strain components can be defined as follows

$$\underline{\varepsilon}_{1}x_1 = \{\ \varepsilon_x\ \}$$

and

$$\varepsilon_x = -y\frac{d^2v}{dx^2} = -\frac{y}{L^2}\frac{d^2v}{d\xi^2}$$

From $\quad v(\xi) = \sum_{i=1}^{4} \delta_i\ N_i$

it can be shown that

$$v(\xi) = \underline{N}^t\ \underline{\delta}$$

where $\quad \underline{N} = \{\ N_1\quad N_2\quad N_3\quad N_4\ \}$

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nce,

$$\underline{\varepsilon} = -\frac{y}{L^2} \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} \underline{\delta}$$

where $$N_i'' = \frac{d^2N_i}{dx^2}$$

or,

$$\underline{\varepsilon} = \underline{B}_{1X4} \; \underline{\delta}$$

where

$$\underline{B} = -\frac{y}{L^2} \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix}$$

Step 4

Express the strain components at the point in terms of nodal displacements.

The relevant component is σ_x, i.e.

$$\underline{\sigma}_{1X1} = \{\sigma_x\}$$

$$= \underline{D} \; \underline{\varepsilon}$$

where

$$\underline{D}_{1X1} = [E]$$

Hence,

$$\underline{\sigma} = \underline{D} \; \underline{B} \; \underline{\delta}$$

Step 5

Express the total potential energy of the element in terms of nodal displacements.

(i)
$$U = \frac{1}{2} \iiint_{\text{element}} \underline{\varepsilon}^t \, \underline{\sigma} \, dxdydz$$

$$= \frac{1}{2} \iiint \underline{\delta}^t \, \underline{B}^t \, \underline{D} \, \underline{B} \, \underline{\delta} \, dxdydz$$

$$= \frac{1}{2} \, \underline{\delta}^t \left(\iiint \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta}$$

(ii)
$$W = \underline{\delta}^t \, \underline{F}$$

(iii)
$$\chi = U - W$$

$$= \frac{1}{2} \, \underline{\delta}^t \left(\iiint \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta} - \underline{\delta}^t \, \underline{F}$$

Step 6

Apply the minimum total potential energy theorem.

$$\frac{\partial \chi}{\partial \underline{\delta}} = 0$$

Using the given matrix theorems, it can be shown that

$$\left(\iiint \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz \right) \underline{\delta} - \underline{F} = 0$$

or

$$\underline{K}^{(e)} \, \underline{\delta} = \underline{F}$$

where

$$\underline{K}^{(e)} = \iiint_{\text{element}} \underline{B}^t \, \underline{D} \, \underline{B} \, dxdydz$$

10/11/86

.3 NUMERICAL INTEGRATION is the approximate computation of an integral using numerical techniques.

A numerical value for an integration can be estimated by employing a quadrature formula, such that

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} W_i\, f(x_i)$$

Known function

where n is the number of quadrature points, x_i and W_i are the values of x and the weight coefficient at the i^{th} point respectively.

a) Gaussian Quadrature or Gauss-Legendre quadrature

Since the cost of any finite element solution is influenced by the number of quadrature points, it is economical to employ the formula which leads to sufficiently accurate results with the smallest possible number of points. It has been shown that the best one-dimensional formula is the Gaussian quadrature. Any finite integral expression can be evaluated by means of the Gaussian quadrature technique as follows:

a) Modify the integration limits,

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f(u)\,du$$

b) Evaluate the modified expression

$$\int_{-1}^{1} f(u)\,du = \sum_{i=1}^{n} H_i\, f(u_i) + R_n$$

where

$$x_i = \frac{b-a}{2}\,\overset{u_i}{(i+1)} + a, \text{ and}$$

u_i, H_i are the abscissa and weight coefficient of the i^{th} quadrature point.

The derivation of the Gaussian quadrature can be reviewed in Reference 3.1, where the following results have been obtained:

(i) $u_1\ u_2, .., u_n$

equation $L_n(u) = 0$

(ii) The weight coefficients are given by:

$$H_i = 2/\left[\left[L_n'(u_i)\right]^2 (1-u_i^2)\right]$$

(iii) The remainder term is expressed by:

$$R_n = \frac{2^{2n+1}(n!)^4\ f^{(2n)}(\alpha)}{(2n+1)\ [(2n)!]^3},$$

$$-1 < \alpha < 1$$

Hence, an exact estimation of the given integration will be obtained if $f(u)$ is a polynominal of a degree up to $(2n-1)$

b) Modified Gaussian Quadrature by Dr. A-El-Z

As the Gaussian quadrature is defined within the interval $[-1,1]$, most finite element investigators have employed such an interval for the description of intrinsic coordinates of the n-hypercubic elements. However, the natural interval for the n-simplex elements is $[0,1]$. Hence, it has been decided here to choose the $[0,1]$ interval, for this work, as a unique interval for all of the intrinsic and natural coordinates. In order to achieve that, the original Guassian quadrature has been adapted to suit the $[0,1]$ interval as follows:

Assume that

$$\int_0^1 f(\xi)\, d\xi = \sum_{i=1}^{n} W_i\ f(a_i).$$

Applying the following linear transformation

$$\xi = (u + 1)/2$$

it can be shown that

$$a_i = (1 - u_i)/2$$

$$a_{n-i+1} = 1 - a_i$$

$$W_i = W_{n-i+1} = H_i/2$$

where

$i = 1, 2, \ldots, m$, and

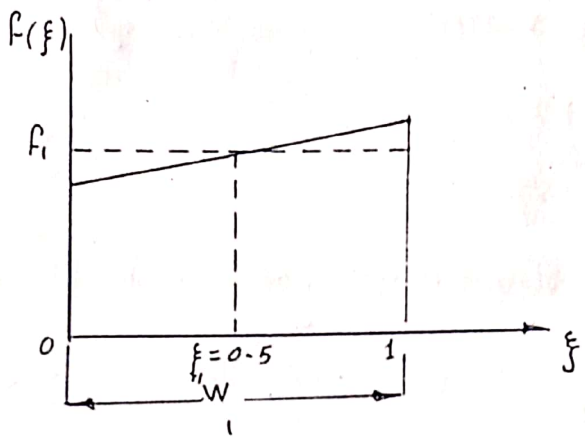
$m = n/2$ if n is even,

$= (n+1)/2$ if n is odd.

The parameters of the modified quadrature have been derived from Gaussian parameters (Ref 3.1) and shown in Table 3.1.

c) Examples

i) One-Point Formula



$n = 1$

$$\int_0^1 f(\xi)\, d\xi = W_1 \; f(\xi_1)$$

where

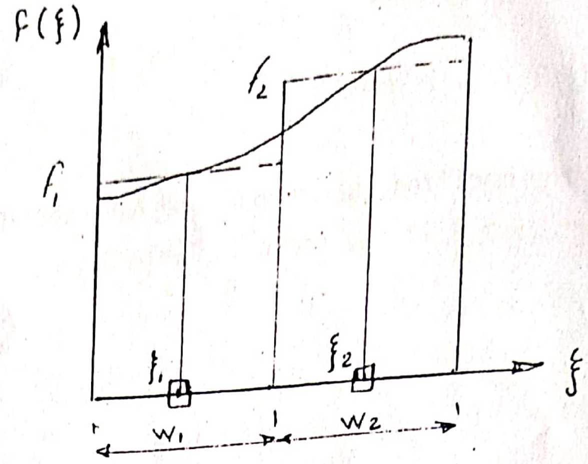
$W_1 = 1.0$

$\xi_1 = 0.5$

as shown in the above figure.

linear, parabolic
cubic

ii) Two-Point Formula



$$\int_0^1 f(\xi)\, d\xi = W_1 \; f(\xi_1) + W_2 \; f(\xi_2)$$

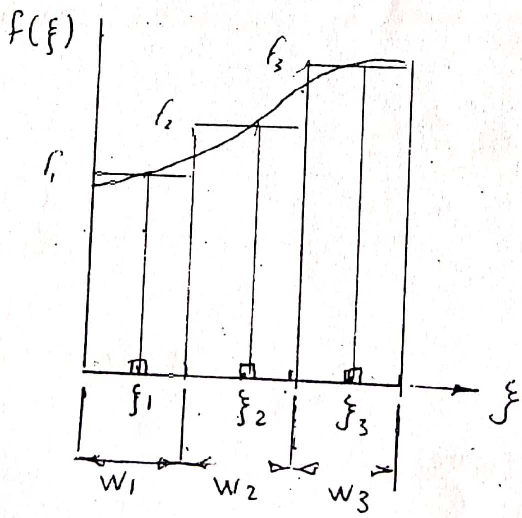
$$W_1 = W_2 = 0.5$$

$$\xi_1 = (3 - \sqrt{3}\,) / 6$$

$$\xi_2 = (3 + \sqrt{3}\,) / 6$$

The exact answer will be obtained if $f(\xi)$ is of degree up to 3.

iii) Three-Point Formula



$$\int_0^1 f(\xi)\, d\xi = W_1 \; f(\xi_1) + W_2 \; f(\xi_2) + W_3 \; f(\xi_3)$$

$$W_1 = 5/18 = W_3$$

$$W_2 = 8/18$$

$$\xi_1 = (5 - \sqrt{15})/10$$

$$\xi_2 = 0.5$$

$$\xi_3 = (5 + \sqrt{15})/10$$

The formula gives the exact answer if $f(\xi)$ is a polynominal of degree up to the fifth.

Exercise 3.2

a) Using Gaussian quadrature, prove that the stiffness matrix for the 2-node beam element is

$$\underline{K}^{(e)} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

b) For the shown cantilever and using the finite element method, find v and θ at the free end.

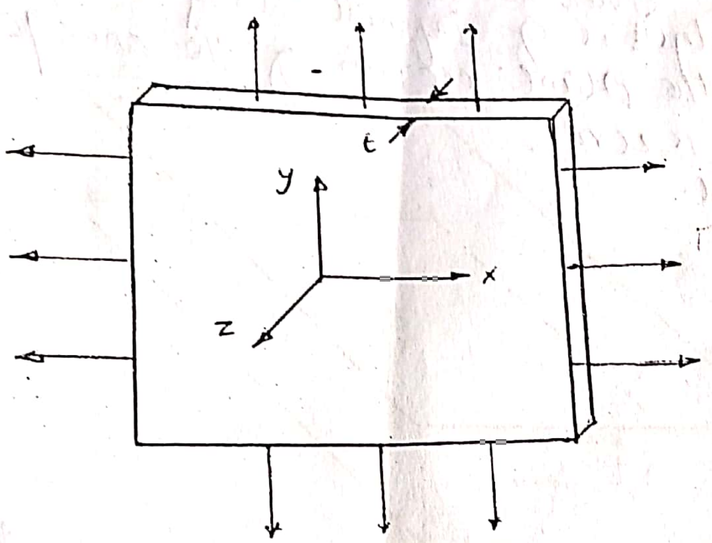
4. TWO DIMENSIONAL ELASTICITY PROBLEMS

4.11.86

4.1 PROBLEM DEFINITION

Generally speaking, most engineering problems are three-dimensional, but by employing some acceptable approximation they may be reduced to one- or two-dimensional situations. A three-dimensional elasticity problem may be approximated into either a plane-stress, or a plane-strain, two dimensional problem as shown below.

a) Plane-Stress Problem



For the case of a thin membrane under in-plane loading, it can be assumed that

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Hence,

$$\gamma_{xz} = \gamma_{yz} = 0$$

$$\varepsilon_z \neq 0$$

$$\sigma_z \varepsilon_z = 0$$

$\varepsilon_z = f(\varepsilon_x, \varepsilon_y)$

The vectors of stress and strain components can be defined as follows

$$\underline{\sigma} = \{ \sigma_x \quad \sigma_y \quad \tau_{xy} \}$$

$$\underline{\varepsilon} = \{ \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \}$$

and the strain energy for a linear elastic material can be expressed as follows

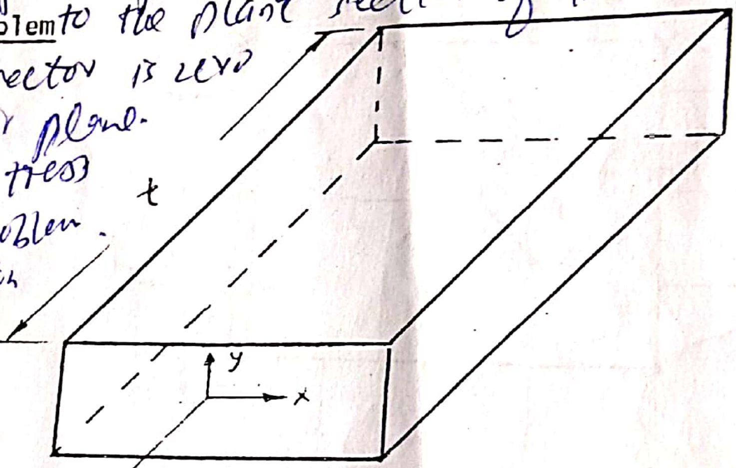
$$U = \tfrac{1}{2} \iiint \underline{\sigma}^t \, \underline{\varepsilon} \; dxdydz$$

$$= \tfrac{1}{2} \iint t \; \underline{\sigma}^t \underline{\varepsilon} \; dxdy$$

plain strain assumes the problem in analysis is of infinite length normal to the plane section of the analysis.

b) <u>Plane-Strain Problem</u>

Plane of the stress vector is zero across a particular plane. It's called plane stress. In plane stress problem the stress variation across thickness is neglected. The plane strain is strain in which all the shape changes of a material happen on a single plane.



For the case of a three-dimensional structure with the following conditions,

i) the x-y plane section is the same at any z, → plane symmetry

ii) there is no force component along the z-axis,

iii) the x-y plane loading is the same at any z, it can be assumed that the state of deformation is the same at any z, i.e.

$$u = u(x, y)$$
$$v = v(x, y)$$
$$w = \text{constant}$$

and $$\frac{\partial}{\partial z}(\text{any property}) = 0$$

Hence,

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

and $\sigma_z \neq 0$ $\quad\quad \sigma_z \varepsilon_z = 0$ $\quad\quad \sigma_z = f(\sigma_x, \sigma_y)$

$$\underline{\sigma} = \{ \sigma_x \ \ \sigma_y \ \ \tau_{xy} \}$$

$$\underline{\varepsilon} = \{ \varepsilon_x \ \ \varepsilon_y \ \ \gamma_{xy} \}$$

$$U = \tfrac{1}{2} \iiint \underline{\sigma}^t \ \underline{\varepsilon} \ dxdydz$$

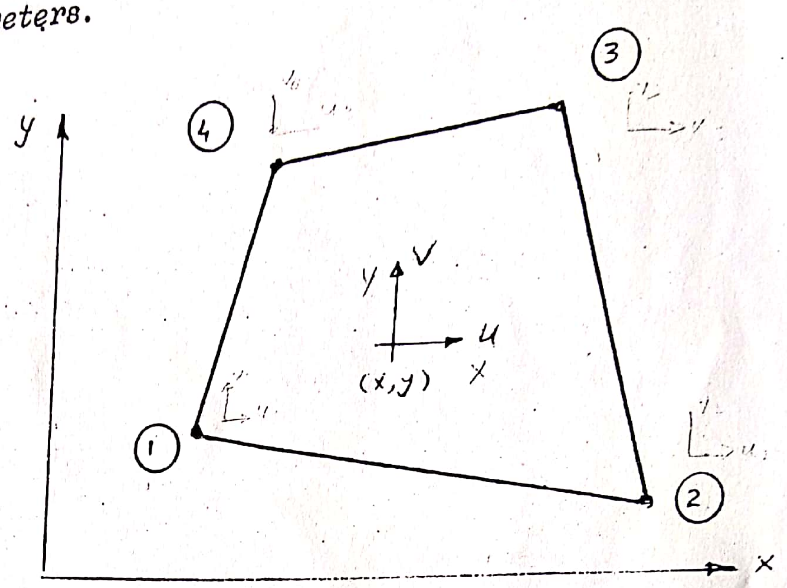
$$= \tfrac{1}{2} \iint t \ \underline{\sigma}^t \ \underline{\varepsilon} \ dxdy$$

where t = element thickness along z-axis.

4.2 DERIVATION OF THE 4-NODE QUADRILATERAL ELEMENT

Step 1

Define the nodal parameters.



The element has 4 corner nodes and 4 straight line sides. Rectangular and parallelogramic elements are special cases. Using a local nodal numbering system, the element can be defined as shown in the above figure. Neglecting the displacement component along the z-axis, the nodal displacement vector can be defined as follows.

Can be global

$$\underline{\delta} = \{ u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \}$$

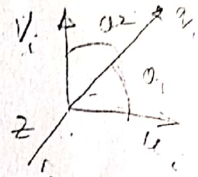
and the nodal force vector is,

$$\underline{F} = \{ F_{x_1} \quad F_{y_1} \quad F_{x_2} \quad F_{y_2} \quad F_{x_3} \quad F_{y_3} \quad F_{x_4} \quad F_{y_4} \}$$

which is equivalent to any real loading system.

Step 2

Express the displacement components at any point in terms of nodal displacements and shape functions.



Define the nodal displacement vector at the i^{th} node as

$$\vec{q}_i = u_i \hat{i} + v_i \hat{j}$$

The displacement vector $\vec{q}$ at any point (x, y) is to be expressed as follows,

$$\vec{q}(x, y) = \sum_{i=1}^{4} \vec{q}_i N_i(x, y)$$

Where

N_1, N_2, N_3, and N_4 are the shape functions.

Hence, it can be shown that

$$u(x, y) = u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4$$

$$v(x, y) = v_1 N_1 + v_2 N_2 + v_3 N_3 + v_4 N_4$$

and it is required to formulate the shape functions.

For the case of the 4-node quadrilateral element, it can be assumed that

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

Substituting these values at the four nodes, it can be shown that

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3$$

$$u_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4$$

or, in matrix form

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

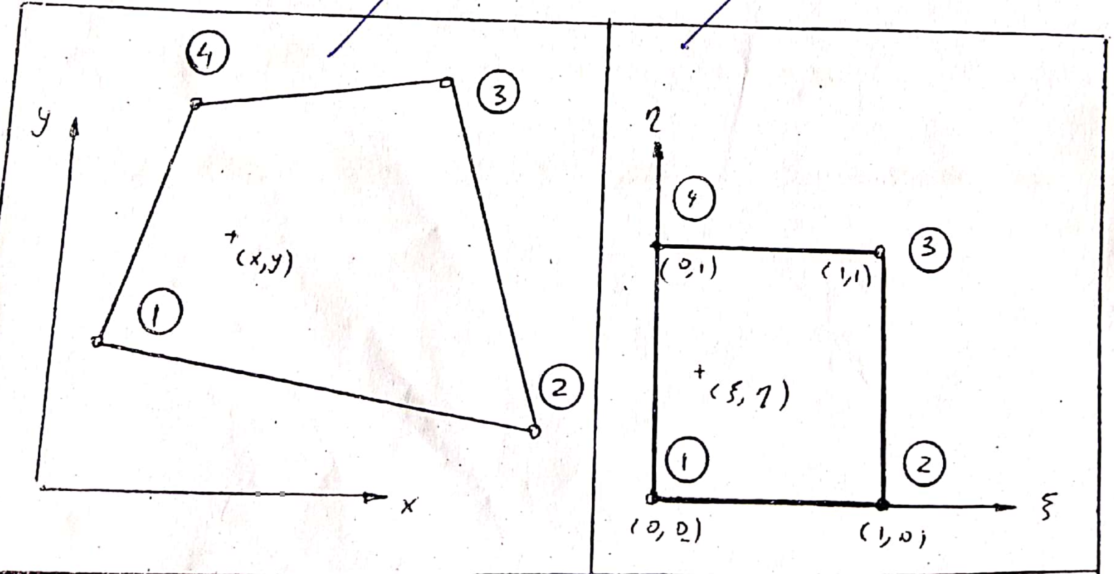
This matrix equation represents 4 simultaneous equations in the 4 unknowns α_1, α_2, α_3 and α_4 which can be solved. By substituting back into the $u(x,y)$ expression, the shape functions can be deduced.

The major difficulty is that the shape functions depend upon the nodal co-ordinates and for every different subdomain the previous procedure of shape function derivation should be repeated.

Imp

It is useful to employ a local system which is simple, unique and independent of the global system. Such a system is known as an intrinsic system and its co-ordinates are the intrinsic co-ordinates ξ and η. Let us discuss the unique intrinsic element in the ξ-η system and then try to correlate the ξ-η system with the x-y system.

One very good idea, is to transform the quadrilateral element into a square of unit side length as shown below. Isoparametric transformation



Element in Cartesian x-y Plane

Element in Intrinsic ξ-η Plane

he rectangular element shape functions can be employed, where

$$N_1 = (1 - \xi)(1 - \eta)$$

$$N_2 = \xi(1 - \eta)$$

$$N_3 = \xi\eta$$

$$N_4 = (1 - \xi)\eta$$

and hence

$$u(\xi, \eta) = \sum_{i=1}^{4} u_i \, N_i(\xi, \eta)$$

$$v(\xi, \eta) = \sum_{i=1}^{4} v_i \, N_i(\xi, \eta)$$

The problem now reduces to one of obtaining the equations of the transformation

$$x = x(\xi, \eta)$$

$$y = y(\xi, \eta)$$

Another valid idea, is to assume that x and y are field functions defined within the element domain in terms of their nodal values and that they obey the same interpolation formula, i.e.

$$x(\xi, \eta) = x_1 N_1 + x_2 N_2 + x_3 N_3 + x_4 N_4$$

$$y(\xi, \eta) = y_1 N_1 + y_2 N_2 + y_3 N_3 + y_4 N_4$$

Such a transformation is known as "Isoparametric Transformation".

under special circumstances the same shape function can also be used to specify the relation b/w the Global and local coordinates system. If this is so the element is of type called Isoparametric – eg 4-side quadrilateral

Finally, the displacement vector $\underline{q}$ at any point (x, y) can be expressed as follows

$$\underline{q} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \underline{\delta}$$

Step 3

Express the strain components at any point in terms of nodal displacements and shape functions.

Defining

$$\underline{\delta} = \{ u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \}$$

it can be shown that

$$u(x,y) = (N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0 \quad N_4 \quad 0) \ \underline{\delta}$$

$$v(x,y) = (0 \quad N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0 \quad N_4) \ \underline{\delta}$$

From strain-displacement relations,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \left(\frac{\partial N_1}{\partial x} \quad 0 \quad \frac{\partial N_2}{\partial x} \quad 0 \quad \frac{\partial N_3}{\partial x} \quad 0 \quad \frac{\partial N_4}{\partial x} \quad 0 \right) \underline{\delta}$$

$$\varepsilon_y = \frac{\partial u}{\partial y} = \left(0 \quad \frac{\partial N_1}{\partial y} \quad 0 \quad \frac{\partial N_2}{\partial y} \quad \cdots \right) \underline{\delta}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial N_1}{\partial y} \quad \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial y} \quad \frac{\partial N_2}{\partial x} \quad \cdots \right) \underline{\delta}$$

ce, it can be deduced that

$$\underset{3\times 1}{\underline{\varepsilon}} = \underset{3\times 8}{\underline{B}} \quad \underset{8\times 1}{\underline{\delta}}$$

where

$$\underline{\varepsilon} = \{ \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots \end{bmatrix}$$

Since the shape functions are expressed in terms of the intrinsic co-ordinates ξ, η it is useful to deduce the Cartesian derivatives in terms of intrinsic derivatives. Consider the i^{th} shape function, and apply the chain rule of partial differentiation.

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}$$

n matrix form

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\[2em] \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \underline{J}\left(\frac{x,y}{\xi,\eta}\right) \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\[2em] \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

where

$$\underline{J}\left(\frac{x,y}{\xi,\eta}\right) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2em] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Imp

which is known as the Jacobian matrix.

Finally,

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\[2em] \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \underline{J}^{-1}\left(\frac{x,y}{\xi,\eta}\right) \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\[2em] \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

and it can be shown that

$$\underline{J}^{-1} = \frac{1}{|\underline{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\[2em] -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

$$\underline{J}^{-1} = \frac{1}{A} \begin{bmatrix} y_4 - y_1 & -(y_2 - y_1) \\ -(x_4 - x_1) & x_2 - x_1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{A} \begin{bmatrix} y_4 - y_1 & -(y_2 - y_1) \\ -(x_4 - x_1) & x_2 - x_1 \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

or explicitly,

$$\frac{\partial N_i}{\partial x} = \frac{1}{A} \left[(y_4 - y_1) \frac{\partial N_i}{\partial \xi} - (y_2 - y_1) \frac{\partial N_i}{\partial \eta} \right]$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{A} \left[-(x_4 - x_1) \frac{\partial N_i}{\partial \xi} + (x_2 - x_1) \frac{\partial N_i}{\partial \eta} \right]$$

Unfortunately, it is not easy to express the Cartesian derivatives of a shape function in terms of its intrinsic derivatives for the general quadrilateral element. A numerical solution should be employed.

Step 4

Express the stress components at any point in terms of nodal displacements and shape functions.

Define $\quad \underline{\sigma} = \{ \sigma_x \quad \sigma_y \quad \tau_{xy} \}$

the absence of initial and thermal strains (and stresses), $\underline{\sigma}$ be expressed in terms of $\underline{\varepsilon}$ for a linear elastic material follows

$$\underset{3\times1}{\underline{\sigma}} = \underset{3\times3}{\underline{D}} \; \underset{3\times1}{\underline{\varepsilon}}$$

Hence,

$$\underline{\sigma} = \underline{D} \; \underline{B} \; \underline{\delta}$$

Stress-Strain Matrix Imp

From theory of elasticity

$$\varepsilon_x = \frac{1}{E} \left(\sigma_x - \nu(\sigma_y + \sigma_z) \right)$$

$$\varepsilon_y = \frac{1}{E} \left(\sigma_y - \nu(\sigma_z + \sigma_x) \right)$$

$$\varepsilon_z = \frac{1}{E} \left(\sigma_z - \nu(\sigma_x + \sigma_y) \right)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E} \tau_{yz}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E} \tau_{zx}$$

a) **Plane Stress** Imp 1985 + Numerical

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0$$

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E} (\sigma_y - \nu\sigma_x)$$

i.e.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}$$

Hence

$$\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} = E \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix}$$

From

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11} a_{22} - a_{12} a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

it can be shown that

$$\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}^{-1} = \frac{1}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

Also,

$$\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}$$

$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}$

Hence, it can be shown that

$$\underline{\sigma} = \underline{D} \ \underline{\varepsilon}$$

Imp

where

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

D matrix for plane stress

b) <u>Plane Strain</u>

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$$

$$\varepsilon_z = 0 = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y) \right)$$

i.e. $\sigma_z = \nu(\sigma_x + \sigma_y)$

Convert σ_z in terms of σ_x, σ_y

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z) \right)$$

$$= \frac{1}{E}\left(\sigma_x - \nu\{ \sigma_y + \nu(\sigma_x + \sigma_y) \} \right)$$

$$= \frac{1}{E}\left((1 - \nu^2)\,\sigma_x - \nu(1 + \nu)\,\sigma_y \right)$$

$$= \left(\frac{1+\nu}{E}\right)\left((1 - \nu)\,\sigma_x - \nu\sigma_y \right)$$

Similarly,

$$\varepsilon_y = \frac{1+\nu}{E}\left((1 - \nu)\,\sigma_y - \nu\sigma_x \right)$$

or

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu \\ -\nu & 1-\nu \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \sigma_x \\ \\ \sigma_y \end{bmatrix} = \frac{E}{1+\nu}\left(\frac{1}{1-2\nu}\right) \begin{bmatrix} 1-\nu & \nu \\ \\ \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \\ \varepsilon_y \end{bmatrix}$$

and it can be shown that

$$\underset{3\times1}{\underline{\sigma}} = \underset{3\times3}{\underline{D}}\ \underset{3\times1}{\underline{\varepsilon}}$$

where

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Imp

$$\underline{\sigma} = \underline{D}\ \underline{B}\ \underline{\varepsilon}$$

Step 5

Express the total potential energy of the element in terms of nodal displacements.

$$\chi = U - W$$

$$U = \tfrac{1}{2} \iiint_{\text{element}} \underline{\sigma}^t \underline{\varepsilon}\ dxdydz$$

$$W = \underline{\delta}^t \underline{F}$$

From the previous steps

$$\underline{\varepsilon} = \underline{B}\ \underline{\delta}$$

$$\underline{\sigma} = \underline{D}\ \underline{B}\ \underline{\delta}$$

$$\underline{\sigma}^t = \underline{\delta}^t\ \underline{B}^t\ \underline{D}^t$$

$$= \underline{\delta}^t\ \underline{B}^t\ \underline{D}$$

($\underline{D}$ is a symmetric matrix, i.e. $\underline{D} = \underline{D}^t$)

Scanned with CamScanner

i.e.,

$$U = \tfrac{1}{2} \iiint \underline{\delta}^t \; \underline{B}^t \; \underline{D} \; \underline{B} \; \underline{\delta} \; dxdydz$$

$$= \tfrac{1}{2} \; \underline{\delta}^t \left(\iiint \underline{B}^t \; \underline{D} \; \underline{B} \; dxdydz \right) \underline{\delta}$$

and

$$\chi = \tfrac{1}{2} \; \underline{\delta}^t \left(\iiint \underline{B}^t \; \underline{D} \; \underline{B} \; dxdydz \right) \underline{\delta} - \underline{\delta}^t \; \underline{F}$$

<u>Step 6</u>

Apply the minimum total potential energy theorem.

$\chi = \min$ $\Rightarrow \frac{\partial \chi}{\partial \delta} = 0$

$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{0} = \left(\iiint \underline{B}^t \; \underline{D} \; \underline{B} \; dxdydz \right) \underline{\delta} - \underline{F}$$

or

$$\underline{K}_{(e)} \; \underline{\delta} = \underline{F}$$

where

$$\underline{K}_{(e)} = \iiint_{element} \underline{B}^t \; \underline{D} \; \underline{B} \; dxdydz$$

since $\underline{B}$ is independent of z, it can be deduced that

$$\underline{K}_{(e)} = \iint_{element} t \; \underline{B}^t \; \underline{D} \; \underline{B} \; dxdy$$

Imp

where t = element thickness along z direction.

i.e.

$$dA = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta$$

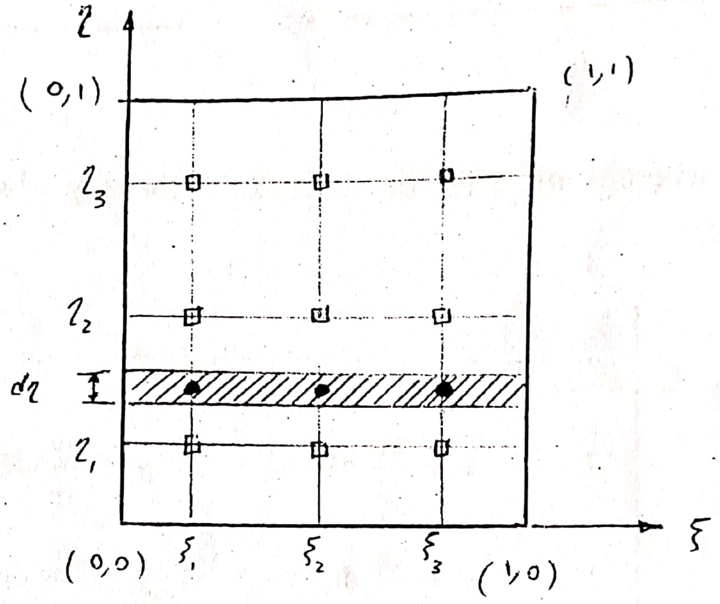
$$= \left| \underline{J} \left(\frac{x,y}{\xi,\eta} \right) \right| d\xi d\eta$$

Hence,

$$\underline{K}_{(e)} = \int_0^1 \int_0^1 t \; \underline{B}^t \; \underline{D} \; \underline{B} \; |\underline{J}| \; d\xi d\eta$$

Double Integration method is a powerful tool in solving deflection and slope of a Beam at any point.

The Double Integration Problem



$$\int_0^1 \int_0^1 f(\xi, \eta) \; d\xi d\eta$$

$$= \int_0^1 \left\{ \int_0^1 f(\xi, \eta) \; \partial\xi \right\} d\eta$$

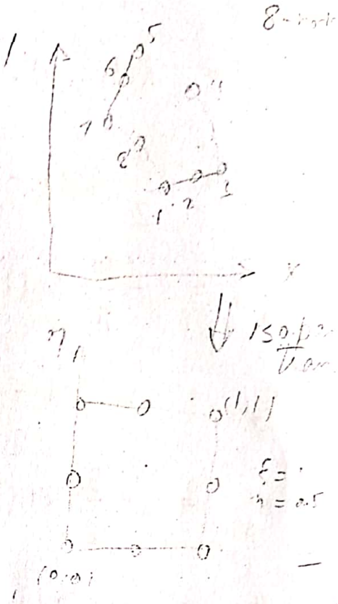
Applying the modified Gaussian quadrature:

$$= \int_0^1 \left\{ \sum_{r=0}^{n} W_r \; f(\xi_r, \eta) \right\} d\eta$$

10/11/86

4.4 SHAPE FUNCTION PROPERTIES OR CONVERGENCE CRITERIA

The shape functions should satisfy certain conditions in order that the finite element solution may converge to an accurate solution.



a) Interpolation Condition

$$N_i\ (\xi_j,\ \eta_j) \ = \ \delta_{ij} \quad \text{(The Kronecker delta)}$$

Physical Interpretation

$$u\ (\xi,\ \eta) \ = \ u_1\ N_1(\xi,\ \eta) \ + \ u_2\ N_2(\xi,\ \eta) \ + \ \ldots$$

At the j^{th} node,

$$u\ (\xi_j,\ \eta_j) \ = \ u_j \ = \ u_1\ N_1(\xi_j,\ \eta_j) \ + \ u_2\ N_2(\xi_j,\ \eta_j) \ + \ \ldots$$

Hence $$N_i\ (\xi_j,\ \eta_j) \ = \ \delta_{ij}$$

$u(\xi_j, \eta_j) = u_j$

b) Rigid Translation Condition (Constant Property condition)

$$\sum_{i=1}^{n} N_i(\xi,\ \eta) \ = \ 1$$

where n = number of element nodes

Physical Interpretation

means that are the ... same distance ... rigidly

If the element is moved rigidly a distance c along the x-direction

$$u_1 \ = \ u_2 \ = \ \ldots \ = \ u_n \ = \ c \ = \ u(\xi,\ \eta)$$

$$u(\xi, \eta) = \sum_{i=1}^{n} c\, N_i = c \sum_{i=1}^{n} N_i(\xi,\eta) \Rightarrow 1$$

$$= c, \text{ if } \sum_{i=1}^{n} N_i = 1$$

c) <u>Rigid Rotation Condition</u>

arbitrary translation

$$\sum_{i=1}^{n} (\alpha + \beta x_i + \gamma y_i)\; N_i(\xi, \eta)$$

$$= \alpha + \beta x + \gamma y$$

<u>Physical Interpretation</u>

Imagine $u_i = \alpha + \beta x_i + \gamma y_i$

Assume that the element is rotated rigidly,

$$u(x, y) = \alpha + \beta x + \gamma y$$

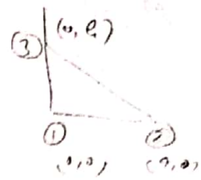
$$u(\xi, \eta) = \sum_{i=1}^{n} u_i\; N_i\,(\xi, \eta)$$

$$= \sum_{i=1}^{n} (\alpha + \beta x_i + \gamma y_i)\; N_i\,(\xi, \eta)$$

<u>Result</u> a + b + c

Any isoparametric element satisfies the rigid rotation condition, so long as

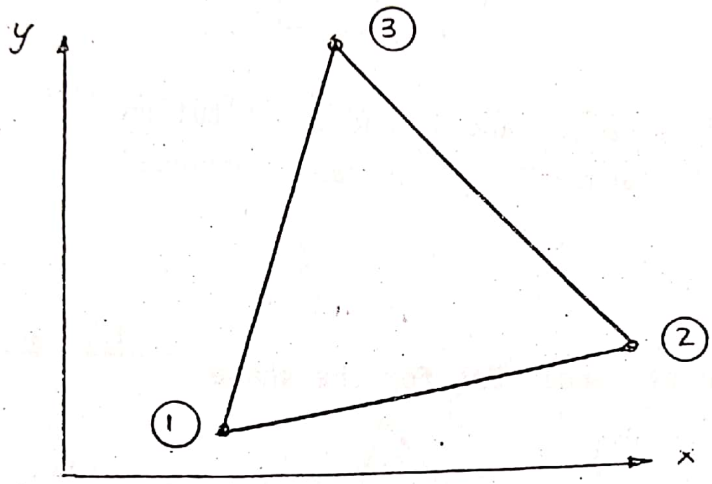
$$\sum_{i=1}^{n} N_i = 1$$



TRIANGULAR ELEMENTS

stead of employing quadrilateral elements, each quadrilateral lement can be replaced by two triangular elements. Triangular lements are very flexible for dealing with complex domains.

a) 3-Node Triangular Element



For the case of a 3-node triangular element, the nodal vectors are defined as follows

$$\underline{\delta} = \{ u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \}$$

$$\underline{F} = \{ F_{x_1} \quad F_{y_1} \quad F_{x_2} \quad F_{y_2} \quad F_{x_3} \quad F_{y_3} \}$$

It is required to derive three shape functions N_1, N_2 and N_3 such that

$$u(x, y) = u_1 N_1 + u_2 N_2 + u_3 N_3$$

$$v(x, y) = v_1 N_1 + u_2 N_2 + v_3 N_3$$

It can be assumed that

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

Hence,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

By solving the above equations for α_1, α_2 and α_3 and substituting back in the $u(x, y)$ equation, the shape functions can be deduced.

Direct Approach

By selecting the three independent conditions for the shape functions as follows

$$N_1 + N_2 + N_3 = 1$$

$$x_1 N_1 + x_2 N_2 + x_3 N_3 = x$$

$$y_1 N_1 + y_2 N_2 + y_3 N_3 = y$$

ie

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

three linear equations in N_1, N_2, N_3 are obtained.

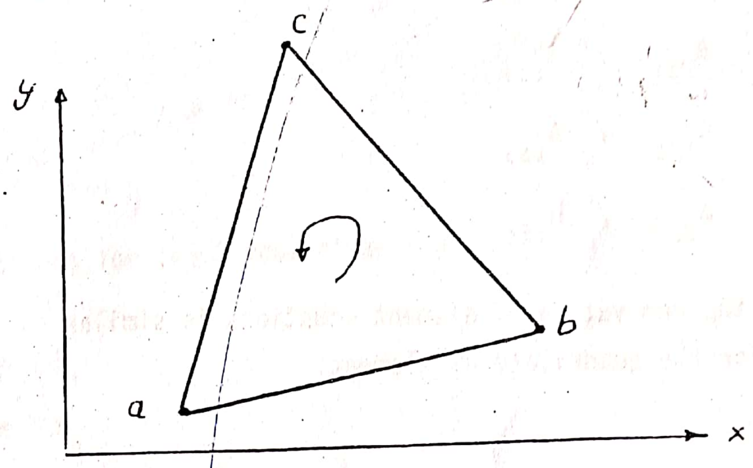
Applying Cramer's rule

$$N_1 = \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$N_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$N_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Geometric interpretation



From analytical geometry, the area of the triangle abc

$$= \Delta abc = \tfrac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{vmatrix}$$

$$= \tfrac{1}{2} \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix}$$

The shape functions of this element are as follows

$$N_1 = \frac{1}{2} L_1 (3L_1 - 1)(3L_1 - 2)$$

$$N_2 = \frac{9}{2} L_1 L_2 (3L_1 - 1)$$

$$N_3 = \frac{9}{2} L_1 L_2 (3L_2 - 1)$$

$$N_4 = \frac{1}{2} L_2 (3L_2 - 1)(3L_2 - 2)$$

$$N_5 = \frac{9}{2} L_2 L_3 (3L_2 - 1)$$

$$N_6 = \frac{9}{2} L_2 L_3 (3L_3 - 1)$$

$$N_7 = \frac{1}{2} L_3 (3L_3 - 1)(3L_3 - 2)$$

$$N_8 = \frac{9}{2} L_3 L_1 (3L_3 - 1)$$

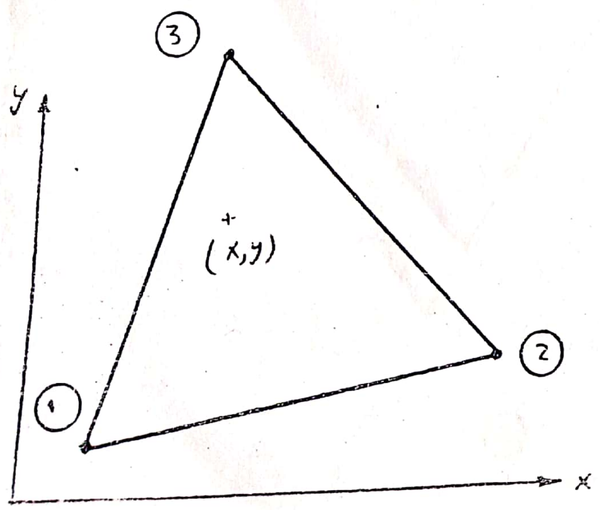
$$N_9 = \frac{9}{2} L_3 L_1 (3L_1 - 1)$$

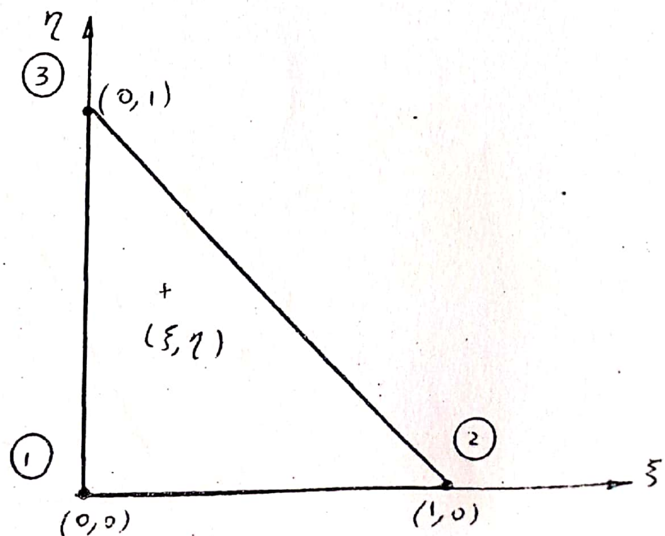
$$N_{10} = 27 L_1 L_2 L_3$$

c) <u>Use of Intrinsic Coordinates (ξ, η)</u>

In order to overcome the difficulties which arise with area coordinates the shape functions can be expressed in terms of intrinsic coordinates.

i) <u>The 3-Node Triangular Element</u>





Element in Intrinsic Plane

nsider the 3-node element shown in the above figure. It can be ssumed that

$$u(\xi, \eta) = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta$$

$$u_1 = \alpha_1$$

$$u_2 = \alpha_1 + \alpha_2$$

$$u_3 = \alpha_1 + \alpha_3$$

Hence,

$$\alpha_1 = u_1$$

$$\alpha_2 = u_2 - u_1$$

$$\alpha_3 = u_3 - u_1$$

and

$$u(\xi, \eta) = u_1 + \xi(u_2 - u_1) + \eta(u_3 - u_1)$$

$$= u_1(1 - \xi - \eta) + u_2 \xi + u_3 \eta$$

Comparing this result with the formula, $u = L_1 u_1 + L_2 u_2 + L_3 u_3$

it can be deduced that

$$L_1 = 1 - \xi - \eta$$

$$L_2 = \xi$$

$$L_3 = \eta$$

The best course now is to employ Argyris' theorem for the derivation of shape functions, it is then possible to transform from the L_1-L_2-L_3 system to the ξ-η system.

The use of ξ-η coordinates will allow the employment of similar expressions for the Jacobian matrix and Cartesian derivatives of shape functions. This approach can be extended to the 3-node element as shown below.

Scanned with CamScanner

From isoparametric transformation

$$x = x_1 N_1 + x_2 N_2 + x_3 N_3$$

$$= x_1 (1 - \xi - \eta) + x_2 \xi + x_3 \eta$$

i.e. $$x = x_1 + \xi(x_2 - x_1) + \eta(x_3 - x_1)$$

Similarly
$$y = y_1 + \xi(y_2 - y_1) + \eta(y_3 - y_1)$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \underline{J}\left(\frac{x,y}{\xi,\eta}\right) \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

where

$$\underline{J}\left(\frac{x,y}{\xi,\eta}\right) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$

$$|\underline{J}| = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= 2A$$

where A = area of the element

$$\underline{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_3 - y_1 & -(y_2 - y_1) \\ -(x_3 - x_1) & x_2 - x_1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\[2ex] \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \underline{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\[2ex] \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

<u>Exercise</u> 4.1 ?

" difficulty in numerical integration

Prove that

$$\frac{\partial N_i}{\partial x} = \frac{1}{2A} (y_j - y_k)$$

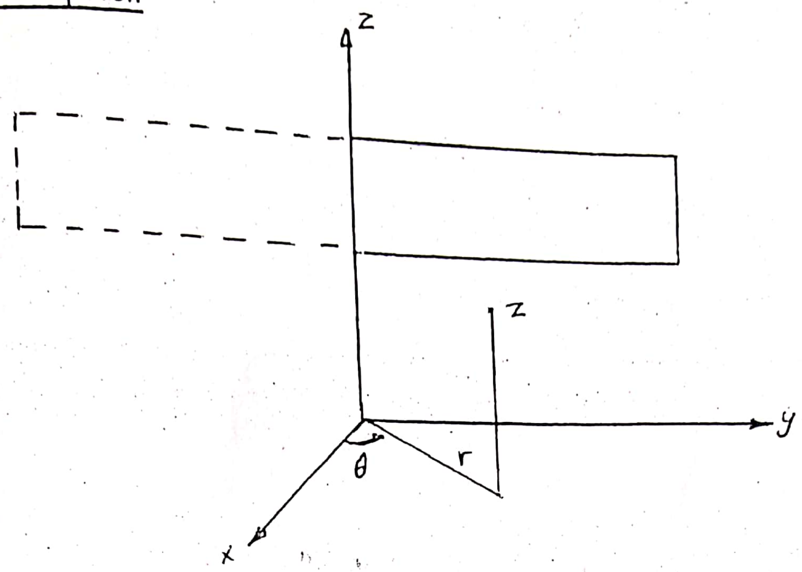
$$\frac{\partial N_i}{\partial y} \overset{?}{=} (-)\frac{1}{2A} (x_j - x_k)$$

where $i, j, k \equiv (1, 2, 3)$ in any circular order.

AXISYMMETRICAL ELASTICITY PROBLEMS

Important

Geometrical Description



Consider any section in the y-z plane. The body generated by rotating such a section, one complete revolution with respect to the z-axis is known as an axisymmetric body. For such a body, with the z-axis as the axis of symmetry, it is useful to use cylindrical coordinates (r, θ, z) to describe any point, where

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = z$$

Conditions for Approximation

In order to approximate the problem to an axisymmetrical elasticity problem the following conditions should be satisfied.

i) The structure is axisymmetric, i.e. the cross-section at any θ should be the same.

ii) There is no loading component normal to the r-z plane.

iii) The load distribution in the r-z plane should be the same at any θ.

the above conditions, the following can be assumed

There is no displacement normal to the r-z plane, i.e.

$$u_\theta = 0$$

$$\frac{\partial(\text{Property})}{\partial\theta} = 0$$ isotropic

i.e. $$\frac{\partial u_r}{\partial\theta} = 0 \qquad \frac{\partial u_z}{\partial\theta} = 0$$

where u_r, u_z, u_θ are the displacement components in the r, z and θ directions.

Following on from the previous conditions and assumptions, it can be seen that the state of deformation and loading is the same at any r-z plane, i.e. the r-z cross-section can represent the whole structure. The finite element mesh can be generated as a two-dimensional mesh in the r-z plane, keeping in mind that the real domain is three-dimensional. Each node is a ring in 3-dimensional space.

Example of the 3-Node Triangular Element

The difference between axisymmetric and two-dimensional elasticity problems can be illustrated through the example of a 3-node triangular element. The steps of the derivation of the element equations will be reviewed in a generalised way.

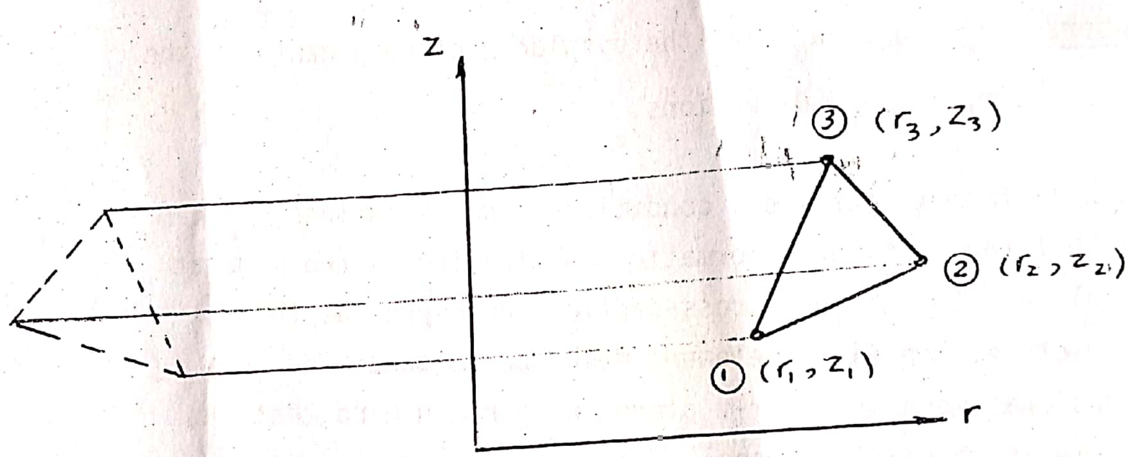
Step 1

Define nodal parameters.

For any n-node element

$$\underline{\delta} = \{ u_{r_1} \quad u_{z_1} \quad u_{r_2} \quad u_{z_2} \quad \cdots \quad u_{r_n} \quad u_{z_n} \}$$

$$\underline{F} = \{ F_{r_1} \quad F_{z_1} \quad F_{r_2} \quad F_{z_2} \quad \cdots \quad F_{r_n} \quad F_{z_n} \}$$



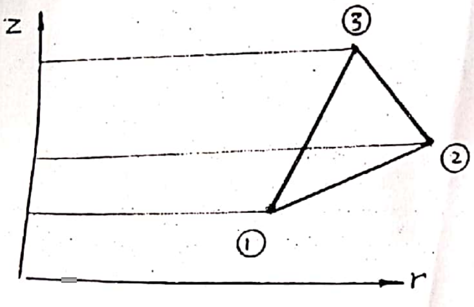
Hence, for the 3-node triangular element, as shown above, the nodal displacement and load vectors are

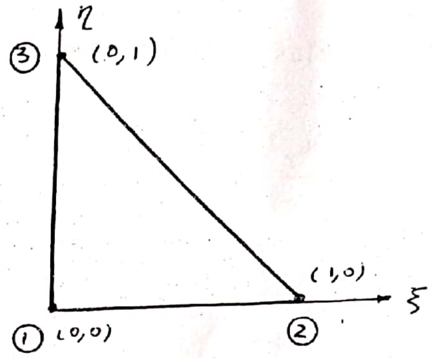
$$\underline{\delta} = \{ u_{r_1} \quad u_{z_1} \quad u_{r_2} \quad u_{z_2} \quad u_{r_3} \quad u_{z_3} \}$$

$$\underline{F} = \{ F_{r_1} \quad F_{z_1} \quad F_{r_2} \quad F_{z_2} \quad F_{r_3} \quad F_{z_3} \}$$

step 2

Express the displacement components at any point in terms of nodal displacements and shape functions.





Using intrinsic coordinates

$$u_r(\xi, \eta) = \sum_{i=1}^{n} (u_r)_i \; N_i(\xi, \eta)$$

$$u_z(\xi, \eta) = \sum_{i=1}^{n} (u_z)_i \; N_i(\xi, \eta)$$

For the 3-node element

$$u_r = (N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0) \; \underline{\delta}$$

$$u_z = (0 \quad N_1 \quad 0 \quad N_2 \quad 0 \quad N_3) \; \underline{\delta}$$

where

$$N_1 = 1 - \xi - \eta$$

$$N_2 = \xi$$

$$N_3 = \eta$$

Step 3

Express the strain components at any point in terms of nodal displacements.

For axisymmetrical elasticity problems:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_\theta = \frac{u_r}{r}$$

$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

$$\gamma_{r\theta} = \gamma_{z\theta} = 0$$

Hence, for the 3-node element

$$\varepsilon_r = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \end{bmatrix} \underline{\delta}$$

$$\varepsilon_z = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \end{bmatrix} \underline{\delta}$$

$$\varepsilon_\theta = \begin{bmatrix} \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix} \underline{\delta}$$

$$\gamma_{rz} = \begin{bmatrix} \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \underline{\delta}$$

It can be shown that

$$\underline{\varepsilon} = \underline{B}\ \underline{\delta}$$

where

$$\underline{\varepsilon} = \{\ \varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz}\ \}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

The problem which now arises is the formulation of the derivatives with respect to r and z. Using the chain rule of partial differentiation, it can be deduced that

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \underline{J}\left(\frac{r,z}{\xi,\eta}\right) \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial z} \end{bmatrix}$$

where

$$\underline{J}\left(\frac{r,z}{\xi,\eta}\right) \quad \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}$$

and

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial r} \\ \\ \dfrac{\partial N_i}{\partial z} \end{bmatrix} = \underline{J}^{-1}\left(\frac{r,z}{\xi,\eta}\right) \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

For the 3-node triangular element, applying the isoparametric transformation:

$$r(\xi, \eta) = r_1 N_1 + r_2 N_2 + r_3 N_3$$

$$= r_1 + \xi(r_2 - r_1) + \eta(r_3 - r_1)$$

$$z(\xi, \eta) = z_1 + \xi(z_2 - z_1) + \eta(z_3 - z_1)$$

$$\underline{J}\left(\frac{r,z}{\xi,\eta}\right) = \begin{bmatrix} r_2 - r_1 & z_2 - z_1 \\ \\ r_3 - r_1 & z_3 - z_1 \end{bmatrix}$$

$|\underline{J}| = 2A$, where A = area of the element,

$$\underline{J}^{-1}\left(\frac{r,z}{\xi,\eta}\right) = \frac{1}{2A} \begin{bmatrix} z_3 - z_1 & -(z_2 - z_1) \\ \\ -(r_3 - r_1) & r_2 - r_1 \end{bmatrix}$$

Step 4

Express the stress components at any point in terms of nodal displacements.

From

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu(\sigma_z + \sigma_\theta) \right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_r + \sigma_\theta) \right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z) \right)$$

$$\gamma_{rz} = \frac{2(1 + \nu)}{E} \tau_{rz}$$

it can be deduced that

$$\underline{\sigma} = \underline{D}\ \underline{\varepsilon}$$

where

$$\underline{\sigma} = \{ \sigma_r \quad \sigma_z \quad \sigma_\theta \quad \tau_{rz} \}$$

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Hence,

$$\underline{\sigma} = \underline{D}\ \underline{B}\ \underline{\delta}$$

<u>Step 5</u>

Express the total potential energy of the element in terms of nodal displacements.

$$\chi = U - W$$

$$W = \underline{\delta}^t \, \underline{F}$$

$$U = \tfrac{1}{2} \iiint_{element} \underline{\sigma}^t \, \underline{\varepsilon} \; d(Vol.)$$

$$= \tfrac{1}{2} \iiint \underline{\delta}^t \, \underline{B}^t \, \underline{D} \, \underline{B} \, \underline{\delta} \; d(Vol.)$$

$$= \tfrac{1}{2} \underline{\delta}^t \left(\iiint \underline{B}^t \, \underline{D} \, \underline{B} \; d(Vol.) \right) \underline{\delta}$$

Step 6

Minimise the total potential energy of the element.

$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{0}$$

$$\left(\iiint \underline{B}^t \, \underline{D} \, \underline{B} \; d(Vol.) \right) \underline{\delta} - \underline{F} = \underline{0}$$

i.e. $\quad \underline{K}_{(e)} \, \underline{\delta} = \underline{F}$

where

$$\underline{K}_{(e)} = \iiint_{element} \underline{B}^t \, \underline{D} \, \underline{B} \; d(Vol.)$$

$d(Vol.) = dxdydz$ in the x-y-z system.

From the previous analysis:

$$dxdy = \left| \underline{J} \left(\frac{x,y}{r,\theta} \right) \right| dr d\theta$$

$$x = r \cos \theta$$
$$y = r \sin \theta$$

$$\underline{J}\left(\frac{x,y}{r,\theta}\right) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

$$\left|\underline{J}\left(\frac{x,y}{r,\theta}\right)\right| = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence,

$$d(\text{Vol.}) = r\,dr\,d\theta\,dz$$

$$K_{(e)} = \iiint \underline{B}^t \, \underline{D} \, \underline{B} \; r\,dr\,d\theta\,dz$$

since the integrand does not vary with θ

$$\int_{\text{element}} d\theta = 2\pi$$

and

$$\underline{K}_{(e)} = \iint_{\substack{\text{cross}\\ \text{section}}} \underline{B}^t \, \underline{D} \, \underline{B} \; 2\pi r \; dr\,dz$$

Also,

$$dr\,dz = \left|\underline{J}\left(\frac{r,z}{\xi,\eta}\right)\right| d\xi\,d\eta$$

i.e.

$$\underline{K}_{(e)} = \iint_{\substack{\text{intrinsic}\\ \text{element}}} \underline{B}^t \, \underline{D} \, \underline{B} \; 2\pi r \; \left|\underline{J}\left(\frac{r,z}{\xi,\eta}\right)\right| d\xi\,d\eta$$

Having discussed the numerical evaluation of double integrand over the quadrilateral domain it is necessary now to extend the approach to the triangular domain.