Vector Integration

This chapter treats integration in vector fields. It is the mathematics that engineers and physicists use to describe fluid flow, design underwater transmission cable, explain the flow of heat in stars, and put satellites in orbit. In particular, we define line integral, which are used to find the work done by a force field in moving an object along a path through the field. We also define surface integrals so we can find the rate that a fluid flows across a surface. Along the way we develop key concepts and result, such as conservative force fields and Green's theorem, to simplify our calculations of these new integrals by connecting them to the single, double, and triple integrals we have already studied.

Objectives :

At the end of this unit he will be able to understand :

- Line integral of a vector function are helpful in determining total work done by a force F & find the circulation of F.
- Greens theorem establishes a connection between a double integral and line integral & Greens theorem in a plane applies to simply connected region bounded by closed curve
- Greens theorem can also be extended to line integrals in space
- Stokes theorem relates the line integral of a vector function to the surface integral of the curl of the vector function
- Stokes theorem is useful in transforming a line integral in to surface integral and vice versa
- Greens theorem in a plane is a special case of stokes theorem .
- The significance of the Divergence theorem lies in the fact that a surface integral may be expressed as a Volume integral and a vice versa

Line integral: Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along a curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

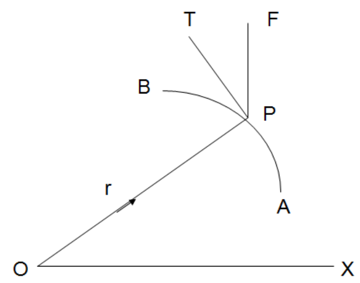
Component of $\vec{F}$ along a tangent PT at P.

$$= \text{Dot product of } \vec{F} \text{ and unit vector along PT}$$

$$= \vec{F} \bullet \frac{\vec{dr}}{ds} \left(\frac{\vec{dr}}{ds} \textit{ is a unit vector along PT} \right)$$

Line Integral $= \sum \vec{F} \bullet \frac{\vec{dr}}{ds}$ *from A to B along the curve*

Therefore Line integral $= \int_c \left(\vec{F} \bullet \frac{\vec{dr}}{ds} \right) ds = \int_c \vec{F} \bullet \vec{dr}$



Note:

(1) **Work**. If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$$= \int_c \vec{F} \bullet \overrightarrow{dr}$$

(2) **Circulation:** If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V} \bullet \overrightarrow{dr}$ is called the circulation of V round the closed curve c.
If the circulation of V round every closed curve is zero then V is said to be irrotational there.

(3) When the path of integration is a closed curve then the notation is $\oint$ in place of $\int$.

Examples:

1. If a force $\vec{F} = 2x^2 y\,\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

Solution:

$$work\ \ done = \int_c \vec{F} \bullet \overrightarrow{dr}$$

$$\vec{r} = x\hat{i} + y\hat{j} \qquad \therefore \quad \overrightarrow{dr} = dx\hat{i} + dy\hat{j}$$

$$= \int_c \left(2x^2 y\hat{i} + 3xy\hat{j}\right) \bullet \left(dx\hat{i} + dy\hat{j}\right)$$

$$= \int_c \left(2x^2 ydx + 3xydy\right)$$

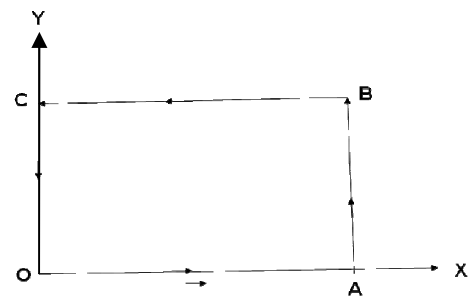
putting the value $y = 4x^2$ *and* $dy = 8xdx$, *we get*

$$= \int_0^1 \left[2x^2\left(4x^2\right)dx + 3x\left(4x^2\right)8xdx\right]$$

$$= 104 \int_0^1 x^4 dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

2. Evaluate $\int_c \vec{F} \bullet \vec{dr}$ $\quad where\ \vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane z=0 and bounded by the lines x=0, y=0, x=a and y=a.

Solution: From the figure ,we have



$$\int_c \vec{F} \bullet \vec{dr} = \int_{OA} \vec{F} \bullet \vec{dr} + \int_{AB} \vec{F} \bullet \vec{dr} + \int_{BC} \vec{F} \bullet \vec{dr} + \int_{CO} \vec{F} \bullet \vec{dr}$$

$Here \quad \vec{r} = x\hat{i} + y\hat{j} \qquad \therefore \quad \vec{dr} = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{i} + xy\hat{j}$

$$\vec{F} \bullet \vec{dr} = x^2 dx + xydy$$

On OA, y=0, $\qquad \therefore \quad \vec{F} \bullet \vec{dr} = x^2 dx$

$$\int_{OA} \vec{F} \bullet \vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB, x=a, $\quad$ dx=0 $\qquad \therefore \quad \vec{F} \bullet \vec{dr} = aydy$

$$\therefore \quad \vec{F} \bullet \vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

On BC , y=a, $\quad$ dy=0 $\qquad \therefore \quad \vec{F} \bullet \vec{dr} = x^2 dx$

$$\int_{BC} \vec{F} \bullet \vec{dr} = \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

On CO , x=0, $\therefore \quad \overrightarrow{F} \bullet \overrightarrow{dr} = 0$

$$\int_{CO} \overrightarrow{F} \bullet \overrightarrow{dr} = 0$$

$$\int_c \overrightarrow{F} \bullet \overrightarrow{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}.$$

3. A vector field is given $\overrightarrow{F} = (2y+3)\hat{i} + (xz)\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_c \overrightarrow{F} \bullet \overrightarrow{dr}$

Along the path c is $x = 2t,\ y = t,\ z = t^3$ from t=0 to t=1.

Solution: we have by definition

$$\int_c \overrightarrow{F} \bullet \overrightarrow{dr} = \int_c [(2x+3)dx + (xz)dy + (yz-x)dz]$$

$$= \int_0^1 [(2t+3)(2dt) + (2t)(t^3 dt) + (t^4 - 2t)(3t^2 dt)]$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1$$

$$= \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= \left[2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}\right] = 7.32857$$

4. If $\overrightarrow{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\int_c \overrightarrow{F} \times \overrightarrow{dr}$ along the curve

$x = \cos t,\ y = \sin t,\ z = 2\cos t$ from t=0 to $t = \frac{\pi}{2}$.

Solution: We have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \therefore \quad \overrightarrow{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\overrightarrow{F} \times \overrightarrow{dr} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= (-zdz - xdy)\hat{i} - (2ydz - xdx)\hat{j} + (2ydy + zdx)\hat{k}$$

$$= [-2\cos t(-2\sin t)dt - \cos t(\cos t)dt]\hat{i}$$
$$-[2\sin t(-2\sin t)dt - \cos t(-\sin t)]\hat{j}$$
$$+[2\sin t(\cos t)dt + 2\cos t(-\sin t)dt]\hat{k}$$

$$= \left(4\cos t\sin t - \cos^2 t\right)\hat{i} - \left(4\sin^2 t - \cos t\sin t\right)\hat{j}$$

$$\therefore \int_c \vec{F}\times\vec{dr} = \int_0^{\frac{\pi}{2}} \left[\left(4\cos t\sin t - \cos^2 t\right)\hat{i} + \left(4\sin^2 t - \cos t\sin t\right)\hat{j}\right]dt$$

$$= \int_0^{\frac{\pi}{2}} \left\{\left[2\sin 2t - \frac{\cos 2t + 1}{2}\right]\hat{i}\right\}dt + \int_0^{\frac{\pi}{2}} \left[\left(2(1-\cos 2t) - \frac{1}{2}\sin 2t\right)\hat{j}\right]dt$$

$$= \left[-\cos 2t - \frac{1}{4}\sin 2t - \frac{1}{2}t\right]_0^{\frac{\pi}{2}}\hat{i} + \left[2t - \sin 2t + \frac{1}{4}\cos 2t\right]_0^{\frac{\pi}{2}}\hat{j}$$

$$= \left[-\cos\pi - \frac{1}{4}\sin\pi - \frac{1}{2}\left(\frac{\pi}{2}\right) + \cos 0 + \frac{1}{4}\sin 0 + \frac{1}{2}(0)\right]\hat{i}$$
$$+ \left[\pi - \sin\pi + \frac{1}{4}\cos\pi - 0 + \sin 0 - \frac{1}{4}\cos 0\right]\hat{j}$$

$$= \left[1 - 0 - \frac{\pi}{4} + 1 + 0\right]\hat{i} + \left[\pi - 0 - \frac{1}{4} + 0 + \frac{1}{4}\right]\hat{j}$$

$$= \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}$$

5. The acceleration of a particle at time t is given by $\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\hat{j} + 6t\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ be zero at t=0, find $\vec{v}$ and $\vec{r}$ at any point t.

Solution: Here $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$

On integrating, we have

$$\vec{v} = \frac{\vec{dr}}{dt} = \hat{i}\int 18\cos 3t dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t dt$$

$$\vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \hat{c} \qquad (1)$$

$At \quad t=0, \quad \vec{v}=\vec{0}$

$putting \quad t=0 \quad and \quad \vec{v}=\vec{0} \quad , \quad we\ get$

$$\vec{0}=4\hat{j}+\vec{c} \quad \Rightarrow \vec{c}=-4\hat{j}$$

$$\therefore \quad \vec{v}=\frac{\overrightarrow{dr}}{dt}=6\sin 3t\,\hat{i}+4(\cos 2t-1)\hat{j}+3t^2\hat{k}$$

Again integrating , we get

$$\vec{r}=\hat{i}\int 6\sin 3t\ dt+\hat{j}\int 4(\cos 2t-1)dt+\hat{k}\int 3t^2 dt$$

$$\Rightarrow \qquad \vec{r}=-2\cos 3t\,\hat{i}+(2\sin 2t-4t)\hat{j}+t^3\hat{k}+\vec{c} \qquad (2)$$

$At, \quad t=0, \quad \vec{r}=0$

$putting\ t=0 \quad and \quad \vec{r}=0 \quad in \quad (2), \quad we\ get$

$$\therefore \quad \vec{0}=-2\,\hat{i}+\overrightarrow{C_1}\Rightarrow \overrightarrow{C_1}=2\hat{i}$$

Hence

$$\vec{r}=2(1-\cos 3t)\,\hat{i}+(2\sin 2t-2t)\hat{j}+t^3\hat{k}\ .$$

6. If $\vec{A}=(3x^2+6y)\hat{i}-14yz\hat{j}+20xz\hat{k}$, Evaluate the line integral $\oint \vec{A}\bullet\overrightarrow{dr}$ from (0,0,0) to (1,1,1) along the curve C $x=t,\ y=t^2,\ z=t^3$.

Solution: We have

$$\oint \vec{A}\bullet\overrightarrow{dr}=\int_c \left[3x^2+6y)\hat{i}\ --14yz\hat{j}+20xz^2\hat{k}\right]\bullet\left[\hat{i}\ dx+\hat{j}dy+\hat{k}\ dz\right]$$

$$=\int_c \left[(3x^2+6y)dx-14yz\hat{j}+20xz^2dz\right]$$

If $x=t,\ y=t^2,\ z=t^3$, then points (0,0,0) and (1,1,1) correspond to t=0 and t=1 respectively.

$$\text{Now,} \quad \int_c \vec{A}\bullet\overrightarrow{dr}=\int_0^1\left[(3t^2+6t^2)dt-14t^2\,t^3\,d(t^2)+20t(t^3)^2\,d(t^3)\right]$$

$$\int_c \vec{A}\bullet\overrightarrow{dr}=\int_0^1\left[(9t^2)dt-14\,t^5\,2t\ dt+20(t^7)3t^2dt\right]$$

$$\int_c \vec{A}\bullet\overrightarrow{dr}=\left[9\left(\frac{t^3}{3}\right)-28\left(\frac{t^7}{7}\right)+60\left(\frac{t^{10}}{10}\right)\right]_0^1$$

$$= 3 - 4 + 6 = 5$$

7. A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral over a circular path $x^2 + y^2 = a^2$, $z = 0$.

Solution: We have

Work done $= \int_c \vec{F} \bullet \vec{dr}$

$$= \int_c \left[(\sin y)\hat{i} + x(1+\cos y)\hat{j}\right] \bullet \left[dx\hat{i} + dy\hat{j}\right] \qquad \because \quad z = 0 \ \& \ dz = 0$$

$$\Rightarrow \int_c \vec{F} \bullet \vec{dr} = \int_c (\sin y dx + x(1+\cos y)dy) = \int_c (\sin y dx + x\cos y dy + x dy)$$

$$= \int_c d(x\sin y) + \int_c x dy$$

The parametric equation of the given path

$x^2 + y^2 = a^2$ *are* $x = a\cos\theta$, $y = a\sin\theta$. θ var*ies from* 0 *to* 2π

$$\int_c \vec{F} \bullet \vec{dr} = \int_0^{2\pi} d[a\cos\theta \sin(a\sin\theta)] + \int_0^{2\pi} a\cos\theta a\cos\theta \, d\theta$$

$$\int_c \vec{F} \bullet \vec{dr} = \int_0^{2\pi} d[a\cos\theta \sin(a\sin\theta)] + \int_0^{2\pi} a^2\cos^2\theta d\theta$$

$$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 0 + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2} 2\pi \quad = 2\pi \ .$$

8. Evaluate $\iint_s \vec{A} \bullet \vec{n} \, dS$, where $\vec{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and S is the part of the plane 2x + 3y + 6z = 12 included in the first octant.

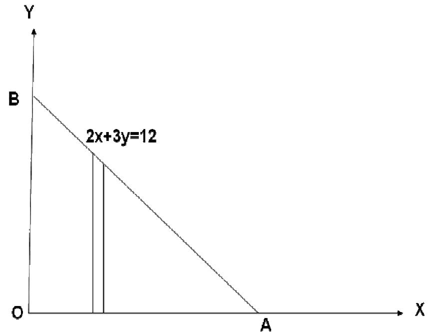
Solution: *Here* $\vec{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$

Given $f(x, y, z) = 2x + 3y + 6z - 12$

Normal vector = $\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n} =$ *Unit normal vector at any po*int (x, y, z) *of* $2x + 3y + 6z = 12$

$$= \frac{\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)}{\sqrt{4+9+36}} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$



$$dS = \frac{dx\,dy}{\hat{n}\bullet\hat{k}} = \frac{dx\,dy}{\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)\bullet\hat{k}} = \frac{dx\,dy}{\frac{6}{7}} = \frac{7}{6}dx\,dy$$

Now,

$$\iint_s \vec{A}\bullet\vec{n}\,dS = \iint\left(18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}\right)\bullet\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)\frac{7}{6}dx\,dy$$

$$= \iint(36z - 36 + 18y)\frac{dx\,dy}{6} = \iint(6z - 6 + 3y)dxdy$$

Putting the value of 6z =12 -12x-3y, we get

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)dxdy$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6-2x)dxdy$$

$$= \int_0^6 (6-2x)dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6-2x)dx \quad [y]_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6-2x)\frac{1}{3}(12-2x)dx$$

$$= \frac{1}{3}\int_0^6 \left(4x^2 - 36x + 72\right)dx$$

$$=\frac{1}{3}\left[\frac{4x^3}{3}-18x^2+72x\right]_0^6$$
$$=\frac{1}{3}[4\times36\times2-18\times36+72\times6]$$
$$=\frac{72}{3}[4-9+6]=24$$

SURFACE INTEGRAL

A surface r = f(u,v) is called smooth if f(u,v) posses continuous first order partial derivative.

Let $\vec{F}$ be a vector function and S be the given surface.

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal = $\vec{F}\bullet\hat{n}$, where n is the unit normal vector to an element ds and

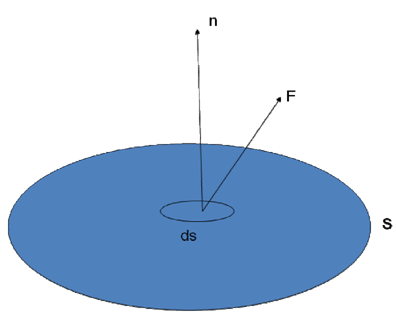
$$\hat{n}=\frac{grad\ f}{|grad\ f|} \qquad ds=\frac{dx\ dy}{(\hat{n}\bullet\hat{k})}$$

Surface integral of F over S

$$\sum\vec{F}\bullet\hat{n}=\iint_s(\vec{F}\bullet\hat{n})ds$$

Note: (1) Flux = $\iint_s(\vec{F}\bullet\hat{n})ds$ where, $\vec{F}$ represents the velocity of a liquid.

If $\iint_s(\vec{F}\bullet\hat{n})ds=0$, then $\vec{F}$ is said to be a solenoidal vector point function.



Example 9. Evaluate $\iint_s \left(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}\right) \bullet ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution: Let $\phi = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\phi = \hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - a^2\right) = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

Given $\vec{F} = \left(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}\right)$

Therefore $\vec{F} \bullet \hat{n} = \left(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}\right) \bullet \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} = \frac{3xyz}{a}$

Now $\iint_s \left(\vec{F} \bullet \hat{n}\right) ds = \iint_s \left(\vec{F} \bullet \hat{n}\right) \frac{dx\,dy}{\left|\hat{k} \bullet \hat{n}\right|} = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{3xyz\,dx\,dy}{a\left(\frac{z}{a}\right)}$

$$= 3\int_0^a \int_0^{\sqrt{a^2 - x^2}} xy\,dy\,dx = 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2 - x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\left(a^2 - x^2\right)dx$$

$$= \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a$$

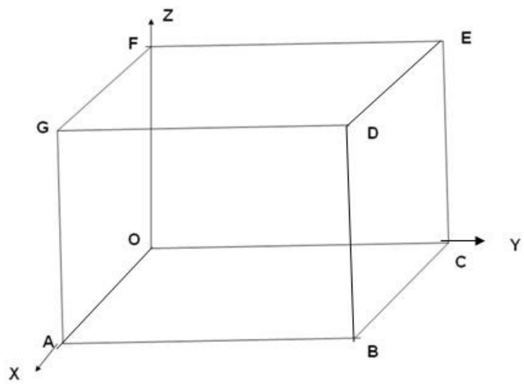
$$= \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{3a^4}{8}$$

Example 10: Show that $\iint_s \left(\vec{F} \bullet \hat{n}\right) ds = \frac{3}{2}$, where $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by the planes,

$$x = 0,\quad x = 1,\quad y = 0,\quad y = 1,\quad z = 0,\quad z = 1$$

Solution:



$$\iint_{s}\left(\vec{F}\bullet\hat{n}\right)ds = \iint_{OABC}\left(\vec{F}\bullet\hat{n}\right)ds + \iint_{DEFG}\left(\vec{F}\bullet\hat{n}\right)ds + \iint_{OAGF}\left(\vec{F}\bullet\hat{n}\right)ds + \iint_{BCED}\left(\vec{F}\bullet\hat{n}\right)ds + \iint_{ABDG}\left(\vec{F}\bullet\hat{n}\right)ds + \iint_{OCEF}\left(\vec{F}\bullet\hat{n}\right)ds$$

But

$$\iint_{OABC}\left(\vec{F}\bullet\hat{n}\right)ds = \iint_{OABC}\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)(-k)dx\,dy = \int_0^1\int_0^1(-yz)dxdy = 0 \quad \because z = 0$$

$$\iint_{DEFG}\left(\vec{F}\bullet\hat{n}\right)ds = \iint_{DEFG}\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\left(\hat{k}\right)dx\,dy = \int_0^1\int_0^1(y\times 1)dxdy$$

$$= \int_0^1 dx\left(\frac{y^2}{2}\right)_0^1 dy = \frac{1}{2}\quad [x]_0^1 = \frac{1}{2}$$

$$\iint_{OAGF}\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\left(-\hat{j}\right)dx\,dz = \int_0^1\int_0^1\left(y^2\right)dxdy = 0 \quad \because y = 0$$

$$\iint_{BCED}\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\left(\hat{j}\right)dx\,dz = \int_0^1\int_0^1\left(-y^2\right)dxdy \quad \because y = 1$$

$$= -\int_0^1 dx\int_0^1 dz = -[x]_0^1\,[z]_0^1 = -1$$

$$\iint_{ABDG}\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)\left(\hat{i}\right)dy\,dz = \int_0^1\int_0^1 4xz\,dydz = \int_0^1\int_0^1 4z\,dydz \quad \because x = 1$$

$$= 4[y]_0^1\left[\frac{z^2}{2}\right]_0^1 = 4\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCEF} \left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)\left(-\hat{i}\right)dy\,dz = \int_0^1\int_0^1 \left(-4xz\right)dydz = 0 \quad \because x = 0$$

On putting al these values, we get

$$\iint_s \left(\vec{F} \bullet \hat{n}\right)ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$

VOLUME INTEGRAL

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_v \vec{F}\,dv$

Example 11: If $\vec{F} = 2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}$, Evaluate $\iiint_v \vec{F}dv$ where, v is the region bounded by the surfaces,

$$x = 0, \quad x = 2, \quad y = 0, \quad y = 4, \quad z = x^2, \quad z = 2.$$

Solution:

$$\iiint_v \vec{F}dv = \iiint \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right)dxdydz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right)dz$$

$$= \int_0^2 dx \int_0^4 dy\, \left[z^2\,\hat{i} - xz\,\hat{j} + yz\,\hat{k}\right]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\, \left[4\hat{i} - 2x\,\hat{j} + 2y\,\hat{k} - x^4\,\hat{i} - x^3\,\hat{j} + x^2 y\,\hat{k}\right]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\,\hat{j} + y^2\,\hat{k} - x^4\,y\,\hat{i} - x^3 y\,\hat{j} + \frac{x^2 y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 \left[16\,\hat{i} - 8x\,\hat{j} + 16\,\hat{k} - 4x^4\,\hat{i} - 4x^3\,\hat{j} - 8\,x^2\hat{k}\right]dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} - x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= \left[32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}\right]$$

$$= \left[\frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}\right] = \frac{32}{15}\left[3\hat{i} + 5\hat{k}\right]$$

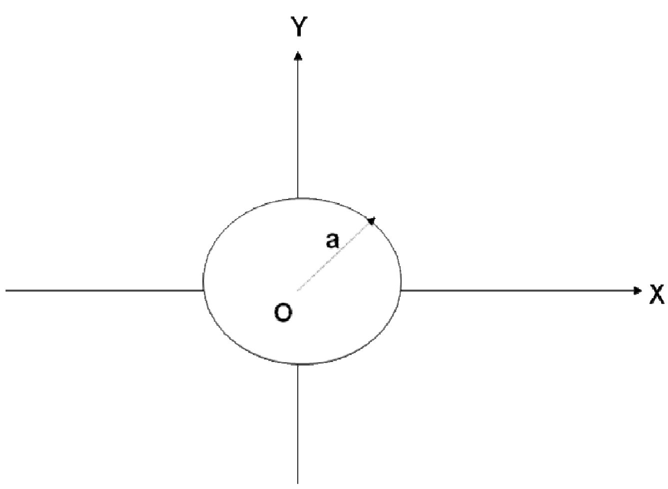
GREEN'S THEOREM

Statement: If $\phi(x, y), \varphi(x, y), \frac{\partial \phi}{\partial y}$ *and* $\frac{\partial \varphi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in xy-plane, then

$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_R \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

Example 12: A vector field $\vec{F}$ is given by $\vec{F} = \sin y\,\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral $\int_C \vec{F} \bullet \vec{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

Solution:



Given $\vec{F} = \sin y\,\hat{i} + x(1 + \cos y)\hat{j}$

$$\int_C \vec{F} \bullet \vec{dr} = [\sin y\,\hat{i} + x(1+\cos y)\hat{j}] \bullet (\hat{i}\ dx + \hat{j}\ dy) = \int_C [\sin y\ dx + x(1+\cos y) dy]$$

On applying Green's Theorem, we have

$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_s \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\ dy$$

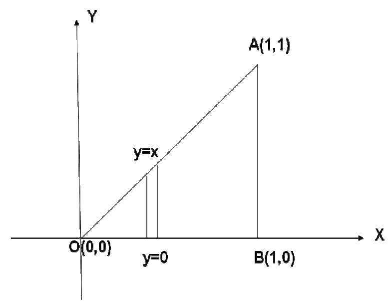
$$= \iint_S [(1+\cos y) - \cos y]\, dx\ dy$$

Where s is the circular plane surface of radius a.

$$= \iint_S dx\ dy = Area\ of\ the\ circle = \pi a^2.$$

Example 13: Using Green's theorem, evaluate $\int_C (x^2\ y\,dx + x^2\,dy)$, where C is the boundary described counter clockwise of the triangle with vertices (0,0),(1,0),(1,1).

Solution: By Green's theorem, we have



$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_s \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\ dy$$

$$\int_C (x^2\ y\,dx + x^2 dy) = \iint_R (2x - x^2)\, dx\ dy$$

$$\int_0^1 (2x - x^2) dx \int_0^x dy = \int_0^1 (2x - x^2) dx [y]_0^x$$

$$= \int_0^1 (2x - x^2) dx [x]$$

$$= \int_0^1 (2x^2 - x^3) dx$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1 = \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}$$

Example 14: Using Green's theorem, evaluate $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region defined by $y = \sqrt{x}$ *and* $y = x^2$.

Solution: Given that $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

By Green's theorem, we have

$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_s \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [(-6y + 16y)] dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [(-6y + 16y)] dx\, dy$$

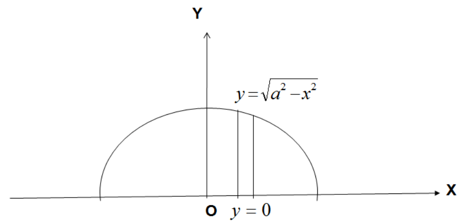
$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \frac{10}{2} \int_0^1 [x - x^4] dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\left(\frac{3}{10}\right) = \frac{3}{2}$$

Example 15: Using Green's theorem, evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the area enclosed by X-axis and the upper half of the circle $x^2 + y^2 = a^2$..

Solution: Given that $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$



By Green's theorem, we have

$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_s \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$= \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \left[\frac{\partial}{\partial x}\left(x^2+y^2\right) - \frac{\partial}{\partial y}\left(2x^2-y^2\right) \right] dx\, dy$$

$$= \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} [(2x+2y)]\, dx\, dy$$

$$= 2\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} [(x+y)]\, dx\, dy$$

$$= 2\int_{-a}^{a} \left(xy + \frac{y^2}{2} \right)_{0}^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left(x\sqrt{a^2-x^2} + \frac{\left(a^2-x^2\right)}{2} \right) dx$$

$$= 2\int_{-a}^{a} x\sqrt{a^2-x^2}\; dx + \int_{-a}^{a} \left(a^2-x^2\right) dx$$

$$= 0 + 2\int_{0}^{a} \left(a^2-x^2\right) = 2\left(a^2 x - \frac{x^3}{3} \right)_{0}^{a}$$

$$= 2\left(a^3 - \frac{a^3}{3} \right) = \frac{4a^3}{3}$$

STOKE'S THEOREM

(Relationship between line integral and surface integral)

Statement: Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Therefore $$\oint_c \vec{F} \bullet d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$$

Where $\hat{n} = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}$ is a unit external normal to any surface ds.

Example 16: Evaluate by Stoke's theorem $\oint_C (yz\,dx + zx\,dy + xy\,dz)$, where C is the curve $x^2 + y^2 = 1, \quad z = y^2$.

Solution : Given $$\oint_C (yz\,dx + zx\,dy + xy\,dz)$$

$$= \int \left(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \oint_c \vec{F} \bullet d\vec{r}$$

$$= \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$$

$$\therefore\ Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

Here $\vec{F} = \left(yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}\right)$

$$= (x - x)\hat{i} + (y - y)\hat{j} + (z - z)\hat{k} = 0$$

$$\therefore\ \oint_c \vec{F} \bullet d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = 0$$

Example 17: Evaluate by Stoke's theorem $\oint_C \left[(2x - y)dx - yz^2 dy - y^2 z dz\right]$, where C is the curve $x^2 + y^2 = 1,$ corresponding to the surface of sphere of unit radius.

Solution : Given $$\oint_C \left[(2x - y)dx - yz^2 dy - y^2 z dz\right]$$

$$= \int \left[(2x - y)\hat{i} + yz\,\hat{j} - y^2 z\hat{k}\right]\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

By Stoke's theorem

$$\oint_c \vec{F} \bullet d\vec{r} = \iint_S curl\ \vec{F} \cdot \hat{n}\ ds$$

$$\therefore \quad Curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\hat{i} + (0 - 0)\hat{j} + (0 + 1)\hat{k} = \hat{k}$$

Therefore

$$\oint_c \hat{k} \cdot \hat{n}\ ds = \iint \hat{k} \cdot \hat{n} \frac{dx\ dy}{\hat{n} \cdot \hat{k}} = \iint dx\ dy = Area\ of\ the\ circle = \pi \qquad \because ds = \frac{dx\ dy}{\hat{n} \cdot \hat{k}}$$

Gauss Divergence Theorem

(Relation between surface integral and volume integral)

Statement: The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically $\quad \iint_S \vec{F} \bullet \hat{n} ds = \iiint_V div\ \overrightarrow{F\ dv}$

Example 1. Evaluate $\iint_S \vec{F} \bullet \hat{n} ds$ using Gauss divergence theorem where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ and $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

Solution: Given $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$ and radius of the sphere r=4

Therefore $\quad \nabla \bullet \vec{F} = \left(\hat{i} \dfrac{\partial}{\partial x} + \hat{j} \dfrac{\partial}{\partial y} + \hat{k} \dfrac{\partial}{\partial z} \right) \bullet (3x\hat{i} + 4y\hat{j} + 5z\hat{k})$

$$\nabla \bullet \vec{F} = 3 + 4 + 5 = 12$$

Then by theorem, we have

$$\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V div\ \overrightarrow{F\ dv} = \iiint_V 12 \bullet dv = 12v$$

Because v is the volume of the sphere

$$= 12 \frac{4}{3} \pi (4)^3 = \frac{3072\pi}{3}$$