① Non-linear Problems

Consider a non-linear B.V.P of the form

$$u''(x) + p(x)u'(x) = f(x, u) \rightarrow ①$$

$$a \le x \le b$$

with the B.cds

$$u'(a) = 0 \ \& \ u(b) = \beta$$

The non-linear term, $f(x,u)$ of the problem needs to be reduced to a sequence of linear problems first before solving the B.V.P. This can be done with the aid of a quasilinerization technique. In this technique, a reasonable initial approx for the function $u(x)$ in $f(x,u)$ is taken, denoted by $u^{(0)}(x)$. Then $f(x,u)$ is expanded around the function $u^{(0)}(x)$ & gives,

$$f(x, u'(x)) = f(x, u^{(0)}(x)) + \left(u'(x) - u^{0}(x)\right)\frac{\partial f}{\partial u}\Big|_{(x,u)} + \cdots$$

In general, we can write as:

②

$$f(x, U^{(s+1)}(x)) = f(x, U^{(s)}(x)) + \left(U^{(s+1)}(x) - U^{(s)}(x)\right) \frac{\partial f}{\partial U}\bigg|_{(x, U^{(s)}(x))} + \cdots$$

$s = 0, 1, 2, \ldots$

where 's' is called iteration index. Hence eq (1) can be approximated as,

$$U_{xx}^{(s+1)}(x) + P(x) U_x^{(s+1)}(x) = f(x, U^{(s)}(x)) + \left(U^{(s+1)}(x) - U^{(s)}(x)\right) \frac{\partial f}{\partial U(x, U^{(s)})}$$

$\Rightarrow$

$$U_{xx}^{(s+1)}(x) + P(x) U_x^{(s+1)}(x) - U^{(s+1)}(x) \frac{\partial f}{\partial U}\bigg|_{(x, U^{(s)}(x))} = f(x, U^{(s)}(x)) - U^{(s)} \frac{\partial f}{\partial U}\bigg|_{(x, U^{(s)}(x))}$$

Therefore, solving the non-linear B.V.P is equivalent to solving the following problem.

$$U_{xx}^{(s+1)}(x) + P(x) U_x^{(s+1)}(x) + q^{(s)}(x) U^{s+1}(x) = r^{(s)}(x) \longrightarrow ②$$

with B-cds $U_x^{(s+1)}(a) = 0$ & $U^{(s+1)}(b) = \beta$.

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where

$$\overset{(s)}{q}(x) = -\left(\frac{\partial f}{\partial u}\right)_{(x,\, \overset{(s)}{u}(x))}$$

$$\overset{(s)}{r}(x) = f(x, \overset{(s)}{u}(x)) - \overset{(s)}{u}(x)\left(\frac{\partial f}{\partial u}\right)_{(x,\, \overset{(s)}{u})}$$

The iteration can be stoped by setting the ~~prescated~~ prescribed absolute error

For example

$$\left| \overset{(s+1)}{u_j} - \overset{(s)}{u_j} \right| \leq 10^{-7}.$$

Example 4.5

Consider the following non-linear BVP

$$u''(x) = -\left(\frac{2}{9}\right) u^{2}(x) \qquad 1 \le x \le 1.5$$

$$u(1) = 1, \quad u(1.5) = (1.5)^{2/3}.$$

The exact solution is $u(x) = x^{2/3}$

Here, the non-linear term is

$$f(x,u) = -\left(\frac{2}{9}\right) u^{2}(x).$$

Thus, the approximation can be obtained by solving the system (2) recursively, with

$$q^{(s)}(x) = -\left(\frac{\partial f}{\partial u}\right)_{(x,\, u^{(s)}(x))} = \frac{4}{9}\, u^{(s)}(x).$$

$$r^{(s)}(x) = f(x, u^{s}(x)) - u^{(s)}\left(\frac{\partial f}{\partial u}\right)_{(x,\, u^{s})}.$$

$$= -\frac{2}{9}\left(u^{(s)}(x)\right)^{2} - u^{s}(x)\left(-\frac{4}{9}\, u^{(s)}(x)\right)$$

$$= \frac{2}{9}\left(u^{(s)}(x)\right)^{2}.$$

The Given Problems becomes as:

$$\left(U''(x)\right)^{(s+1)} + P(x)\left(U'(x)\right)^{(s+1)} + q^{s}(x)\left(U(x)\right)^{s+1} = \gamma^{(s)}(x) \rightarrow (3)$$

$1 \le x \le 1.5$

$U^{(s+1)}(1) = 1, \quad U^{(s+1)}(1.5) = (1.5)^{2/3}.$

where $P(x) = 0.$

$q^{(s)}(x) = \frac{4}{9} U^{(s)}(x)$

$\gamma^{(s)}(x) = \frac{2}{9}\left(U^{(s)}(x)\right)^{2}.$

we consider the initial solutions is $U^{(0)}(x) = 1.$

So

$P(x) = 0.$

$q^{(0)}(x) = \frac{4}{9}.$

$\gamma^{(0)}(x) = \frac{2}{9}$

(3) $\Rightarrow$, $\left(U''(x)\right)^{1} + \frac{4}{9} U^{(1)}(x) = \frac{2}{9}.$

$U^{(1)}(1) = 1, \quad U^{(1)}(1.5) = (1.5)^{2/3}$

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