(1)

Singular BVPs

Consider the following form of singular two-point BVP

$$U''(x) + \frac{c}{x-a} U'(x) + q(x) U(x) = r(x) \quad \rightarrow (1) \quad a \leq x \leq b.$$

with B. cds

$$U'(a) = 0, \ \& \ U(b) = \beta.$$

where C is a constant. It can be noticed that the above (1) second order DE(1) is undefined with $x = a$. Thus, L Hospital rule could be applied at the singular point, to get

$$U''(x) + c\frac{U''(x)}{1} + q(x) U(x) = r(x) \quad \text{for } x = a$$

$$\begin{cases} (1+c) U''(x) + q(a) U(x) = r(a) \\ U''(x) + \frac{c}{x-a} U'(x) + q(x) U(x) = r(x) \quad \text{for } x \neq 0 \end{cases}$$

(2) by using Appraisal

Evaluation at $x=x_j$, gives

$$\begin{cases} (c+1)S'(a) + q(a)S(a) = r(a), & j=0 \\ S''(x_j) + \frac{c}{x_j - a} S'(x_j) + q(x_j)S(x_j) = r(x_j) & j=1,2,\cdots N \end{cases} \longrightarrow (2)$$

along with B. cds.

$$\begin{cases} S'(x_j) = 0 & \text{for } j=0 \\ S(x_j) = \beta & \text{for } j=N \end{cases} \longrightarrow (3)$$

Applying the approximation to the above problem which are given as:

$$\begin{cases} S(x_j) = D_{j-3}\frac{1}{6} + D_{j-2}\frac{2}{3} + D_{j-1}\frac{1}{6}. \\ S'(x_j) = D_{j-3}\left(-\frac{1}{2h}\right) + D_{j-2}(0) + D_{j-1}\left(\frac{1}{2h}\right). \\ S''(x_j) = D_{j-3}\frac{1}{h^2} + D_{j-2}\left(-\frac{2}{h^2}\right) + D_{j-1}\left(\frac{1}{h^2}\right) \end{cases}$$

$$\longrightarrow (4)$$

The equation (2), can be written as:

$$(c+1)\left(D_{-3}\left(-\frac{1}{2h}\right)+D_{-1}\left(\frac{1}{2h}\right)\right)+q(a)\left(\frac{1}{6}D_{-3}+\frac{2}{3}D_{-2}+\frac{1}{6}D_{-1}\right)=r(a).$$

This can be written as:

$$\boxed{\underbrace{\left(-\frac{c+1}{2h}+\frac{q(a)}{6}\right)}_{\alpha}D_{-3}+\underbrace{\frac{2q(a)}{3}}_{\beta}D_{-2}+\overbrace{\left(\frac{c+1}{2h}+\frac{q(a)}{6}\right)}^{\alpha}D_{-1}=r(a)} \quad \rightarrow (5)$$

From (2), we have

$$\underbrace{\left(\frac{1}{h^2}+\frac{c}{x_j-a}\left(-\frac{1}{2h}\right)+\frac{q(x_j)}{6}\right)}_{\alpha_j}D_{j-3}+\underbrace{\left(-\frac{2}{h^2}+\frac{2q(x_j)}{3}\right)}_{\beta_j}D_{j-2}+\underbrace{\left(\frac{1}{h^2}+\frac{c}{x_j-a}\left(\frac{1}{2h}\right)+\frac{2q(x_j)}{3}\right)}_{\alpha_j}D_{j-1}=r(x_j) \quad \rightarrow (6)$$

The B. cds can also be written as:

$$\begin{cases} -\frac{1}{2h} D_{-3} + \frac{1}{2h} D_{-1} = 0 \\ \frac{1}{6} D_{N-3} + \frac{2}{3} D_{N-1} + \frac{1}{6} D_{N-1} = \beta \end{cases} \longrightarrow (7)$$

Equation (5), (6), & (7), can be written in matrix form as:

$$\begin{pmatrix} -\frac{1}{2h} & 0 & \frac{1}{2h} & \cdots & & & 0 \\ \alpha & \beta & \alpha & \cdots & & & 0 \\ \alpha_1 & \beta_1 & \alpha_1 & \cdots & & & 0 \\ 0 & \alpha_2 & \beta_2 & \alpha_2 & \cdots & & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & \alpha_N & \beta_N & \alpha_N & \\ 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{pmatrix} \begin{pmatrix} D_{-3} \\ D_{-2} \\ \vdots \\ D_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma(a) \\ \gamma(x_1) \\ \vdots \\ \gamma(x_N) \\ \beta \end{pmatrix}$$

This system is $N+3 \times N+3$, which provide the values of unknown for the approximation solution: