# Numerical Solution of Third-Order Singularly Perturbed Boundary Value Problems via Quartic Trigonometric B-spline 

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#### Abstract

In this paper, a new quartic trigonometric B-spline is presented to obtain the numerical solutions of third order singularly perturbed boundary value problems in which the highest order derivative is multiplied by a small parameter. The quartic trigonometric B-spline basis function is used and interpolated at the different nodal points. This method is employed after modification of the problem at the point of singularity by L'Hospital rule. Several numerical examples are discussed to exhibit the feasibility and capability of the technique. The maximum errors $\left(L_{\infty}\right)$ and norm errors $\left(L_{2}\right)$ are also computed for different space size steps to assess the performance of the proposed technique. The rate of convergence is second-order. The numerical solutions are compared with both analytical and other existing numerical solutions in the literature. The comparison shows that the new quartic trigonometric B-spline method is more accurate than the ordinary quartic B-spline.


Keywords: Singularly perturbation, two-point boundary value problems, Quartic trigonometric B-spline, Basis function.
Subjclass[2010]: 68W25; 65D05; 65D07; 65L10; 65L11

## 1 Introduction

Singular perturbation problems arise in the modelling of certain phenomen in Engineering and Applied Sciences. The presence of small parameter(s) in these problems sometimes prohibits us from obtaining satisfactory numerical solutions. It is a well-known that often for such problems there are thin layers where the solution varies quickly whilst away from these layers the solution behaves regularly and only varies slowly.

Singular pertubation problems have some major computational difficulties and several methods have been used to provide numerical solutions of these problems. Many authors have proposed their methods for solving singularly perturbation problems. For example, Du [1] has solved a singularly perturbed boundary value problem for nonlinear system using the method of descent and Nagumo conditions. Rang et al.[2] have discussed the singular perturbation of boundary value problem for quasi-linear third order ordinary differential equations (ODE) involving two small parameters using differential inequality method. Pandya and Doctor [3] have suggested a numerical method of finding the numerical solution of third order
singularly perturbed ODE of convection diffusion type. Jang et al [4] have presented a comparison of B-spline method and finite difference method to solve boundary value problems. Cui and Geng [5] have been presented a computational method for solving a third order singularly perturbed boundary value problems using reproducing kernel spaces. Splines of degree four has used to find the approximate solution of a third order self adjoint singularly perturbed boundary value problem by Ghazala[6]. Mishra and Saini $[7,8]$ have considered a third order self adjoint and simple singularly perturbed boundary value problem using Quartic B-spline collocation method. The numerical treatment of singularly perturbed boundary value problems have also been discussed in $[9,10,11,12,13,14,15,16,17]$. In this study, we consider a third-order singularly perturbed boundary value problem

$$
\begin{equation*}
\epsilon y^{\prime \prime \prime}(x)+\frac{\zeta}{x} y^{\prime \prime}(x)+\alpha(x) y^{\prime}(x)+\beta(x) y(x)=\gamma(x) \tag{1.1}
\end{equation*}
$$

with boundary conditions

$$
\begin{equation*}
y(0)=p, y^{\prime}(0)=q, y(1)=r \tag{1.2}
\end{equation*}
$$

where $\zeta, p, q, r$ are finite constants and $\epsilon$ is a small positive parameter such that $0<\epsilon \leq 1$ and $\alpha, \beta, \gamma$ are well ordered, bounded and real functions.
B-spline functions can be used for the numerical solution of linear and nonlinear differential equations due to their important geometric properties and features. Trigonometric B-spline is a non-polynomial B-spline functions containing trigonometric terms. The derivation and properties of trigonometric B-spline can be found in $[18,19]$. In this paper, a new quartic trigonometric B-spline technique is described for the solution of a third order singular boundary value problems. The technique is based on the quartic trigonometric Bspline functions. Some researchers have considered the ordinary B-spline collocation method for solving the proposed problem but, so far as we are aware, not with the new quartic trigonometric B-spline collocation method. The order of convergence can be calculated and was found to be second order. A new quartic trigonometric B-spline is used as an interpolating function in the space dimension. The applicability and accuracy of the technique are demonstrated by applying the scheme to several examples. The numerical results indicate that this method is superior in that it yields more accurate solutions than ordinary quartic B-spline collocation methods [7, 8].

### 1.1 Quartic Trigonometric B-spline functions

The basis of trigonometric B-spline of order 1 can be established from the following formula

$$
T_{i}^{1}(x)= \begin{cases}1 & x \in\left[x_{i}, x_{i+1}\right)  \tag{1.3}\\ 0 & \text { otherwise }\end{cases}
$$

The trigonometric B-spline basis of order $k=2,3, \ldots$ can be obtained by using following recursive formula

$$
\begin{equation*}
T_{i}^{k}=\frac{\sin \left(\frac{x-x_{i}}{2}\right)}{\sin \left(\frac{x_{i+k-1}-x_{i}}{2}\right)} T_{i}^{k-1}(x)+\frac{\sin \left(\frac{x_{i+k}-x}{2}\right)}{\sin \left(\frac{x_{i+k}-x_{i+1}}{2}\right)} T_{i+1}^{k-1}(x) . \tag{1.4}
\end{equation*}
$$

Calculating degree upto 4 for $k=5$, the resulting basis $T_{i}^{5}(x)$ is shown as

$$
T_{i}^{5}(x)=\frac{1}{\phi} \begin{cases}\xi^{4}\left(x_{i}\right), & {\left[x_{i}, x_{i+1}\right]}  \tag{1.5}\\ \xi^{2}\left(x_{i}\right)\left(\xi\left(x_{i}\right) \psi\left(x_{i+2}\right)+\xi\left(x_{i+1}\right) \psi\left(x_{i+3}\right)\right) & \\ +\xi\left(x_{i}\right) \xi^{2}\left(x_{i+1}\right) \psi\left(x_{i+4}\right)+\xi^{3}\left(x_{i+1}\right) \psi\left(x_{i+5}\right), & {\left[x_{i+1}, x_{i+2}\right]} \\ \xi^{2}\left(x_{i}\right) \psi^{2}\left(x_{i+3}\right)+\xi\left(x_{i}\right) \psi\left(x_{i+4}\right)\left(\xi\left(x_{i+1}\right) \psi\left(x_{i+3}\right)\right. & \\ \left.+\psi\left(x_{i+4}\right) \xi\left(x_{i+2}\right)\right)+\psi\left(x_{i+5}\right)\left(\xi^{2}\left(x_{i+1}\right) \psi\left(x_{i+3}\right)\right. & \\ \left.+\psi\left(x_{i+4}\right) \xi\left(x_{i+1}\right) \xi\left(x_{i+2}\right)+\psi\left(x_{i+5}\right) \xi^{2}\left(x_{i+2}\right)\right), & {\left[x_{i+2}, x_{i+3}\right]} \\ \xi\left(x_{i}\right) \psi^{3}\left(x_{i+4}\right)+\psi^{2}\left(x_{i+5}\right)\left(\xi\left(x_{i+2}\right) \psi\left(x_{i+4}\right)\right. & \\ \left.+\xi\left(x_{i+3}\right) \psi\left(x_{i+5}\right)\right)+\xi\left(x_{i+1}\right) \psi^{2}\left(x_{i+4}\right) \psi\left(x_{i+5}\right), & {\left[x_{i+3}, x_{i+4}\right]} \\ \psi^{4}\left(x_{i+5}\right), & {\left[x_{i+4}, x_{i+5}\right]}\end{cases}
$$

where

$$
\xi\left(x_{i}\right)=\sin \left(\frac{x-x_{i}}{2}\right), \psi\left(x_{i}\right)=\sin \left(\frac{x_{i}-x}{2}\right), \phi=\sin \left(\frac{h}{2}\right) \sin (h) \sin \left(\frac{3 h}{2}\right) \sin (2 h)
$$

where $h=(b-a) / n$ and $T_{i}^{5}(x)$ is piecewise trigonometric function of degree 4 with continuity of order 3 . $T_{i}(x), T_{i}^{\prime}(x), T_{i}^{\prime \prime}(x), T_{i}^{\prime \prime \prime}(x)$, are evaluated at different knots which are compiled in the following Table 1 where
$a_{1}=\sin ^{3}\left(\frac{h}{2}\right) \csc (h) \csc \left(\frac{3 h}{2}\right) \csc (2 h), a_{2}=\sin ^{2}\left(\frac{h}{2}\right) \csc (h) \csc (2 h)+2 \sin \left(\frac{h}{2}\right) \sin (h) \csc \left(\frac{3 h}{2}\right) \csc (2 h)$, $a_{3}=\frac{\csc (h) \sec (h)}{2+4 \cos (h)}, a_{4}=\csc (2 h), a_{5}=\csc (h) \csc (2 h), a_{6}=\cos \left(\frac{h}{2}\right)(-4 \cot (h)+\csc (h)) \csc \left(\frac{3 h}{2}\right) \csc (2 h)$,
$a_{7}=\frac{\csc (h)+\csc (2 h)}{1-\cos (h)}, a_{8}=\frac{4-\sec (h)}{\sin (h)+\sin (2 h)-\sin (3 h)}$.

## 2 Description of Quartic Trigonometric B-spline Method (QuTBM)

Consider the third order singularly perturbed boundary value problem (1.1) with small parameter $0<$ $\epsilon \leq 1$. L'Hospital rule is applied to remove the singularity of the given equation at $x=0$. Hence the given equation (1.1) can be changed into the following form

$$
\begin{equation*}
(\epsilon+\zeta) y^{\prime \prime \prime}(x)+\alpha(x) y^{\prime}(x)+\beta(x) y(x)=\gamma(x) \tag{2.1}
\end{equation*}
$$

To illustrate the basic idea of this method, we use the solution domain $a \leq x \leq b$ is partitioned into $n+1$ uniformly spaced points $x_{i}$ such that $0=x_{0}<x_{1}<x_{2}<\ldots<x_{n}=1$. Our perspective for third order singularly perturbed boundary value problems using collocation method with quartic trigonometric B-spline is to consider an approximate solution as

$$
\begin{equation*}
S_{T}(x)=\sum_{i=-4}^{n-1} c_{i} T_{i}^{5}(x) \tag{2.2}
\end{equation*}
$$

where $c_{i}$ are the real non-zero coefficients. Using the table of trigonometric B-spline function, we get approximate values of $y\left(x_{i}\right), y^{\prime}\left(x_{i}\right), y^{\prime \prime}\left(x_{i}\right), y^{\prime \prime \prime}\left(x_{i}\right)$ as

$$
\begin{aligned}
y\left(x_{i}\right) & \approx S_{T}\left(x_{i}\right)=c_{i-4} a_{1}+c_{i-3} a_{2}+c_{i-2} a_{2}+c_{i-1} a_{1} \\
y^{\prime}\left(x_{i}\right) & \approx S_{T}^{\prime}\left(x_{i}\right)=c_{i-4}\left(-a_{3}\right)+c_{i-3}\left(-a_{4}\right)+c_{i-2} a_{4}+c_{i-1} a_{3} \\
y^{\prime \prime}\left(x_{i}\right) & \approx S_{T}^{\prime \prime}\left(x_{i}\right)=c_{i-4} a_{5}+c_{i-3}\left(-a_{5}\right)+c_{i-2}\left(-a_{5}\right)+c_{i-1} a_{5} \\
y^{\prime \prime \prime}\left(x_{i}\right) & \approx S_{T}\left(x_{i}\right)=c_{i-4} a_{6}+c_{i-3} a_{7}+c_{i-2}\left(-a_{7}\right)+c_{i-1} a_{8}
\end{aligned}
$$

After substituting the equation (2.2) into equation (1.1), for $i=1,2, \ldots, n$ it becomes

$$
\begin{equation*}
\epsilon S_{T}^{\prime \prime \prime}\left(x_{i}\right)+\frac{\zeta}{x_{i}} S_{T}^{\prime \prime}\left(x_{i}\right)+\alpha\left(x_{i}\right) S_{T}^{\prime}\left(x_{i}\right)+\beta\left(x_{i}\right) S_{T}\left(x_{i}\right)=\gamma\left(x_{i}\right), \tag{2.3}
\end{equation*}
$$

and for $i=0$ equation (2.1) becomes

$$
(\epsilon+\zeta) S_{T}^{\prime \prime \prime}\left(x_{i}\right)+\alpha\left(x_{i}\right) S_{T}^{\prime}\left(x_{i}\right)+\beta\left(x_{i}\right) S_{T}\left(x_{i}\right)=\gamma\left(x_{i}\right)
$$

with the boundary conditions

$$
\begin{aligned}
S_{T}\left(x_{i}\right) & =p, i=0 \\
S_{T}^{\prime}\left(x_{i}\right) & =q, i=0 \\
S_{T}\left(x_{i}\right) & =r, i=n .
\end{aligned}
$$

A system of $(n+4)$ linear equations with $(n+4)$ unknown is thus obtained. Now, this system of equations can be written as

$$
A X_{n}=B_{n}
$$

where $X_{n}=\left[c_{-4}, c_{-3}, \ldots, c_{n-1}\right]^{T}, B_{n}=\left[p, q, \gamma\left(x_{0}\right), \ldots, \gamma\left(x_{n}\right), r\right]^{T}$ and $A$ is an $(n+4) \times(n+4)$ dimensional matrix. This means that there are infinitely many solutions for $c_{-4}, c_{-3}, \ldots, c_{n-1}$.

## 3 Numerical Examples and Discussions

In this section, three examples are executed to demonstrate the efficiency of the proposed method. Results established by quartic trigonometric B-spline are found to be a good agreement with the exact solutions. To test the accuracy of proposed method, the maximum error $\left(L_{\infty}\right)$ is calculated using the following formula [20, 21, 22]

$$
L_{\infty}=\left\|S_{T}\left(x_{i}\right)-y\left(x_{i}\right)\right\|_{\infty}=\max \left|S_{T}\left(x_{i}\right)-y\left(x_{i}\right)\right| .
$$

The order of convergence R of the proposed method, is determined by the following formula [20, 21, 22]

$$
R=\frac{\log \left(L_{\infty}\left(n_{i}\right)\right)-\log \left(L_{\infty}\left(n_{i+1}\right)\right)}{\log \left(n_{i+1}\right)-\log \left(n_{i}\right)} .
$$

Problem 3.1. Consider the singularly perturbed ODE [8]

$$
\epsilon y^{\prime \prime \prime}(x)+\frac{1}{x} y^{\prime \prime}(x)+y(x)=\gamma(x)
$$

with boundary conditions

$$
y(0)=0, y^{\prime}(0)=9 \epsilon, y(1)=3 \epsilon \sin 3,
$$

where

$$
\gamma(x)=3 \epsilon\left[\sin 3 x-27 \epsilon \cos 3 x-\frac{9}{x} \sin 3 x\right] .
$$

The analytic solution of the given problem is $y=3 \epsilon \sin 3 x$. For the purpose of comparison, we record the absolute errors $\left(L_{\infty}\right)$ with several space step size and different values of parameter $\epsilon$ in Table 2 and compared with quartic B-spline method [8]. The numerical results obtained in this paper are found to be more accurate in comparison to quartic B-spline method [8]. The Table 3 shows that the method has a second order of convergence. we observed from this table the order of convergence has increased when the parameter $\epsilon \rightarrow 0$. Figure 1 depicts the comparison of approximate solution with exact solution with $h=\frac{1}{16}$ and $\epsilon=2^{-4}$.

Problem 3.2. Consider the third order singularly perturbed differential equation [8]

$$
\epsilon y^{\prime \prime \prime}(x)+\frac{5}{x} y^{\prime \prime}(x)+8 y^{\prime}(x)+3 y(x)=\gamma(x)
$$

with boundary conditions

$$
y(0)=0, y(1)=0, y^{\prime}(0)=0,
$$

where

$$
\gamma(x)=-3 x^{3}-21 x^{2}+16 x-6 \epsilon+\frac{10}{x}-30 .
$$

The analytical solution is $y(x)=x^{2}-x^{3}$. In Table 4 we show the absolute error obtained for this problem at different values of parameter $\epsilon$ with different space step size. They indicate that numerical results of the proposed method are in good agreement with given exact solutions and more accurate as compared to QuBM [8]. Numerical results in Table 5 indicate that the computational order of convergence of our new trigonometric method with different space step size, is exactly equal to two. Figure 2 depicts the comparison of exact and approximate solution of this problem at $h=\frac{1}{16}$ and $\epsilon=2^{-4}$.

Problem 3.3. Consider the third order singularly perturbed differential equation [8]

$$
\epsilon y^{\prime \prime \prime}(x)+\frac{2}{x} y^{\prime \prime}(x)+y^{\prime}(x)+y(x)=\gamma(x)
$$

with boundary conditions

$$
y(0)=0, y(1)=1, y^{\prime}(0)=\frac{\frac{1}{\sqrt{\epsilon}}}{\sin \left(\frac{1}{\sqrt{\epsilon}}\right)}
$$

where

$$
\gamma(x)=\frac{\sin \left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin \left(\frac{1}{\sqrt{\epsilon}}\right)}\left[1-\frac{2}{\epsilon x}\right] .
$$

The exact solution of problem-3 is $y(x)=\frac{\sin \left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin \left(\frac{1}{\sqrt{\epsilon}}\right)}$. In Tables 6 and 7 we report the absolute errors of this problem using present method at $\epsilon=10^{-3}$ with different space step size. They also indicate that the order of convergence of the cubic trigonometric B-spline method presented in these tables is equal to two. Figure 3 illustrates the comparison of exact solution with approximate solution of this problem for $\epsilon=10^{-2}$ when $h=\frac{1}{16}$.

## 4 Concluding remarks

In this paper a numerical approach based on new quartic trigonometric B-spline functions has been used to solve third order singular boundary value problems. The new quartic trigonometric B-spline method used in this paper is simple and straight forward to apply. The numerical results reported in Tables 2-6
and depicted in the graphs illustrate the applicability and accuracy of the method when compared with other available methods such as the quartic B-spline collocation methods [8]. The order of convergence is shown to be approximately equal to two.

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Table 1: Values of $T_{i}^{5}(x)$ and its derivatives at different knots.

| $\mathrm{T}(\mathrm{x})$ | $x_{i}$ | $x_{i+1}$ | $x_{i+2}$ | $x_{i+3}$ | $x_{i+4}$ | $x_{i+5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{i}(x)$ | 0 | $a_{1}$ | $a_{2}$ | $a_{2}$ | $a_{1}$ | 0 |
| $T_{i}^{\prime}(x)$ | 0 | $-a_{3}$ | $-a_{4}$ | $a_{4}$ | $a_{3}$ | 0 |
| $T_{i}^{\prime \prime}(x)$ | 0 | $a_{5}$ | $-a_{5}$ | $-a_{5}$ | $a_{5}$ | 0 |
| $T_{i}^{\prime \prime \prime}(x)$ | 0 | $a_{6}$ | $a_{7}$ | $-a_{7}$ | $a_{8}$ | 0 |

Table 2: Comparison of the absolute errors $\left(L_{\infty}\right)$ of present method between QuBM [8] at different values of $n$ and parameter $\epsilon$ for Problem 3.1

| $\epsilon$ | QuTBM $(n=16)$ | QuTBM $(n=32)$ | QuTBM $(n=64)$ | QuTBM $(n=128)$ |
| :---: | :---: | :---: | :---: | :---: |
| $2^{-4}$ | $6.96 \mathrm{E}-06$ | $1.69 \mathrm{E}-06$ | $4.18 \mathrm{E}-07$ | $1.04 \mathrm{E}-07$ |
| $2^{-6}$ | $6.27 \mathrm{E}-07$ | $1.40 \mathrm{E}-07$ | $3.40 \mathrm{E}-08$ | $8.05 \mathrm{E}-09$ |
| $2^{-8}$ | $5.87 \mathrm{E}-08$ | $1.03 \mathrm{E}-08$ | $2.32 \mathrm{E}-09$ | $5.66 \mathrm{E}-10$ |
| $2^{-10}$ | $8.78 \mathrm{E}-09$ | $9.29 \mathrm{E}-10$ | $1.63 \mathrm{E}-10$ | $3.69 \mathrm{E}-11$ |
| $2^{-12}$ | $1.91 \mathrm{E}-09$ | $1.37 \mathrm{E}-10$ | $1.45 \mathrm{E}-11$ | $2.56 \mathrm{E}-12$ |
| $2^{-14}$ | $4.63 \mathrm{E}-10$ | $2.98 \mathrm{E}-11$ | $2.15 \mathrm{E}-12$ | $2.27 \mathrm{E}-13$ |
| $2^{-16}$ | $1.15 \mathrm{E}-10$ | $7.20 \mathrm{E}-12$ | $4.65 \mathrm{E}-13$ | $3.35 \mathrm{E}-14$ |
| $2^{-18}$ | $2.87 \mathrm{E}-11$ | $1.79 \mathrm{E}-12$ | $1.12 \mathrm{E}-13$ | $2.26 \mathrm{E}-15$ |
| $2^{-20}$ | $7.17 \mathrm{E}-12$ | $4.46 \mathrm{E}-13$ | $2.79 \mathrm{E}-14$ | $1.76 \mathrm{E}-15$ |
| $\epsilon$ | $Q u B M[8](n=16)$ | QuBM[8] $(n=32)$ | $Q u B M[8](n=64)$ | $Q u B M[8](n=128)$ |
| $2^{-4}$ | $1.41 \mathrm{E}-05$ | $3.42 \mathrm{E}-06$ | $8.47 \mathrm{E}-07$ | $2.11 \mathrm{E}-07$ |
| $2^{-6}$ | $1.27 \mathrm{E}-06$ | $2.84 \mathrm{E}-07$ | $6.89 \mathrm{E}-08$ | $1.71 \mathrm{E}-08$ |
| $2^{-8}$ | $1.19 \mathrm{E}-07$ | $2.09 \mathrm{E}-08$ | $4.71 \mathrm{E}-09$ | $1.15 \mathrm{E}-09$ |
| $2^{-10}$ | $1.78 \mathrm{E}-08$ | $1.88 \mathrm{E}-09$ | $3.31 \mathrm{E}-10$ | $7.46 \mathrm{E}-11$ |
| $2^{-12}$ | $3.86 \mathrm{E}-09$ | $2.78 \mathrm{E}-10$ | $2.94 \mathrm{E}-11$ | $5.18 \mathrm{E}-12$ |
| $2^{-14}$ | $9.37 \mathrm{E}-10$ | $6.03 \mathrm{E}-11$ | $4.34 \mathrm{E}-12$ | $4.60 \mathrm{E}-13$ |
| $2^{-16}$ | $2.33 \mathrm{E}-10$ | $1.46 \mathrm{E}-11$ | $9.42 \mathrm{E}-13$ | $6.78 \mathrm{E}-14$ |
| $2^{-18}$ | $5.80 \mathrm{E}-11$ | $3.62 \mathrm{E}-12$ | $2.28 \mathrm{E}-13$ | $1.47 \mathrm{E}-14$ |
| $2^{-20}$ | $1.45 \mathrm{E}-11$ | $9.03 \mathrm{E}-13$ | $5.65 \mathrm{E}-14$ | $3.56 \mathrm{E}-15$ |

Table 3: The order of convergence at different values of $n$ for Problem 3.1

| R | Order of Convergence |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $\epsilon=2^{-4}$ | $\epsilon=2^{-6}$ | $\epsilon=2^{-8}$ | $\epsilon=2^{-10}$ | $\epsilon=2^{-12}$ | $\epsilon=2^{-14}$ | $\epsilon=2^{-16}$ | $\epsilon=2^{-18}$ | $\epsilon=2^{-20}$ |  |
| $n=16$ | $\ldots \ldots$ | $\ldots .$. | $\ldots \ldots$ | $\ldots .$. | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots .$. | $\ldots .$. | $\ldots .$. |  |
| $n=32$ | 2.04255 | 2.16107 | 2.50894 | 3.24050 | 3.79704 | 3.95796 | 3.99565 | 4.00378 | 4.00583 |  |
| $n=64$ | 2.01436 | 2.04166 | 2.14851 | 2.50897 | 3.24009 | 3.79583 | 3.95315 | 3.99015 | 3.99912 |  |
| $n=128$ | 2.00225 | 2.00935 | 2.03794 | 2.14607 | 2.50444 | 3.23972 | 3.79442 | 3.95276 | 3.98909 |  |

Table 4: Comparison of the absolute errors $\left(L_{\infty}\right)$ of present method between QuBM [8] at different values of $n$ and parameter $\epsilon$ for Problem 3.2

| $\epsilon$ | QuTBM $(n=16)$ | QuTBM $(n=32)$ | QuTBM $(n=64)$ | QuTBM $(n=128)$ |
| :---: | :---: | :---: | :---: | :---: |
| $2^{-4}$ | $7.42 \mathrm{E}-06$ | $1.87 \mathrm{E}-06$ | $4.70 \mathrm{E}-07$ | $1.18 \mathrm{E}-07$ |
| $2^{-6}$ | $1.84 \mathrm{E}-06$ | $4.69 \mathrm{E}-07$ | $1.18 \mathrm{E}-07$ | $2.96 \mathrm{E}-08$ |
| $2^{-8}$ | $4.35 \mathrm{E}-07$ | $1.16 \mathrm{E}-07$ | $2.94 \mathrm{E}-08$ | $7.40 \mathrm{E}-09$ |
| $2^{-10}$ | $8.29 \mathrm{E}-08$ | $2.75 \mathrm{E}-08$ | $7.28 \mathrm{E}-09$ | $1.84 \mathrm{E}-09$ |
| $2^{-12}$ | $2.90 \mathrm{E}-08$ | $5.46 \mathrm{E}-09$ | $1.73 \mathrm{E}-09$ | $4.56 \mathrm{E}-10$ |
| $2^{-14}$ | $4.13 \mathrm{E}-08$ | $1.14 \mathrm{E}-09$ | $3.50 \mathrm{E}-10$ | $1.09 \mathrm{E}-10$ |
| $2^{-16}$ | $4.55 \mathrm{E}-08$ | $1.99 \mathrm{E}-09$ | $5.07 \mathrm{E}-11$ | $2.22 \mathrm{E}-11$ |
| $2^{-18}$ | $4.66 \mathrm{E}-08$ | $2.27 \mathrm{E}-09$ | $1.08 \mathrm{E}-10$ | $2.56 \mathrm{E}-12$ |
| $2^{-20}$ | $4.68 \mathrm{E}-08$ | $2.34 \mathrm{E}-09$ | $1.26 \mathrm{E}-10$ | $6.30 \mathrm{E}-12$ |
| $\epsilon$ | QuBM[8] $(n=16)$ | QuBM[8](n=32) | QuBM[8] $(n=64)$ | QuBM[8] $n=128)$ |
| $2^{-4}$ | 1.7329 | 1.6937 | 1.7904 | 1.8385 |
| $2^{-6}$ | 0.1142 | 2.0667 | 1.6590 | 1.6841 |
| $2^{-8}$ | 0.0321 | 0.0563 | 3.1185 | 1.6592 |
| $2^{-10}$ | 0.0236 | 0.0165 | 0.0281 | 0.0233 |
| $2^{-12}$ | 0.0219 | 0.0122 | 0.0084 | 0.0140 |
| $2^{-14}$ | 0.0215 | 0.0113 | 0.0062 | 0.0042 |
| $2^{-16}$ | 0.0214 | 0.0111 | 0.0057 | 0.0031 |
| $2^{-18}$ | 0.0213 | 0.0110 | 0.0056 | 0.0029 |
| $2^{-20}$ | 0.0213 | 0.0110 | 0.0056 | 0.0028 |

Table 5: The order of convergence at different values of $n$ for Problem 3.2

| R | Order of Convergence |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $\epsilon=2^{-4}$ | $\epsilon=2^{-6}$ | $\epsilon=2^{-8}$ | $\epsilon=2^{-10}$ | $\epsilon=2^{-12}$ | $\epsilon=2^{-14}$ | $\epsilon=2^{-16}$ | $\epsilon=2^{-18}$ |  |
| $\epsilon=2^{-20}$ |  |  |  |  |  |  |  |  |  |
| $n=16$ | $\ldots .$. | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots \ldots$ | $\ldots \ldots$ |  |
| $n=32$ | 1.98545 | 1.97167 | 1.90606 | 1.99120 | 2.41214 | 5.17787 | 4.51587 | 4.35721 |  |
| $n=64$ | 1.99520 | 1.99208 | 1.97743 | 1.91936 | 1.95405 | 1.90463 | 5.29553 | 4.39197 |  |
| $n=128$ | 1.99711 | 1.99629 | 1.99317 | 1.98025 | 1.92696 | 1.98689 | 1.99269 | 5.40437 |  |
| $n=4.32534$ |  |  |  |  |  |  |  |  |  |

Table 6: The absolute error $\left(L_{\infty}\right)$ and order of convergence at different step size for $\epsilon=10^{-3}$ for Problem 3.3

| $h=\frac{1}{n}$ | QuBM $[8]$ | QuTBM | Order of Convergence |
| :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | $2.10 \mathrm{E}-02$ | $3.27 \mathrm{E}-03$ | $\ldots .$. |
| $\frac{1}{32}$ | $6.59 \mathrm{E}-03$ | $7.77 \mathrm{E}-04$ | 2.07223 |
| $\frac{1}{64}$ | $1.51 \mathrm{E}-03$ | $1.97 \mathrm{E}-04$ | 1.97729 |
| $\frac{1}{128}$ | $3.77 \mathrm{E}-04$ | $4.96 \mathrm{E}-05$ | 1.99296 |
| $\frac{1}{256}$ | $9.55 \mathrm{E}-05$ | $1.24 \mathrm{E}-05$ | 1.99868 |
| $\frac{1}{512}$ | $2.41 \mathrm{E}-05$ | $3.07 \mathrm{E}-06$ | 2.01335 |
| $\frac{1}{1024}$ | $6.08 \mathrm{E}-06$ | $5.18 \mathrm{E}-07$ | 2.56865 |

Table 7: The absolute error $\left(L_{\infty}\right)$ and order of convergence at different step size for $\epsilon=10^{-3}$ for Problem 3.3

| $h=\frac{1}{n}$ | QuBM $[8]$ | QuTBM | Order of Convergence |
| :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | $4.26 \mathrm{E}-01$ | $4.20 \mathrm{E}-01$ | $\ldots \ldots$ |
| $\frac{1}{32}$ | $3.81 \mathrm{E}-02$ | $2.20 \mathrm{E}-02$ | 4.25432 |
| $\frac{1}{64}$ | $4.34 \mathrm{E}-02$ | $3.03 \mathrm{E}-03$ | 2.86067 |
| $\frac{1}{128}$ | $7.18 \mathrm{E}-03$ | $6.87 \mathrm{E}-04$ | 2.13871 |
| $\frac{1}{256}$ | $1.13 \mathrm{E}-03$ | $1.73 \mathrm{E}-04$ | 1.99308 |
| $\frac{1}{512}$ | $2.42 \mathrm{E}-04$ | $4.33 \mathrm{E}-05$ | 1.99570 |
| $\frac{1}{1024}$ | $5.84 \mathrm{E}-05$ | $1.08 \mathrm{E}-05$ | 2.00300 |



Figure 1: Comparison graph of approximated solution and exact solution for Problem 3.1 when $h=\frac{1}{16}$ and $\epsilon=2^{-4}$


Figure 2: Comparison graph of approximated solution and exact solution for Problem 3.2 when $h=\frac{1}{16}$ and $\epsilon=2^{-4}$.


Figure 3: Comparison graph of approximate solution and exact solution for $\epsilon=10^{-2}$ when $h=\frac{1}{16}$.

