The Finite Square Well Potential:

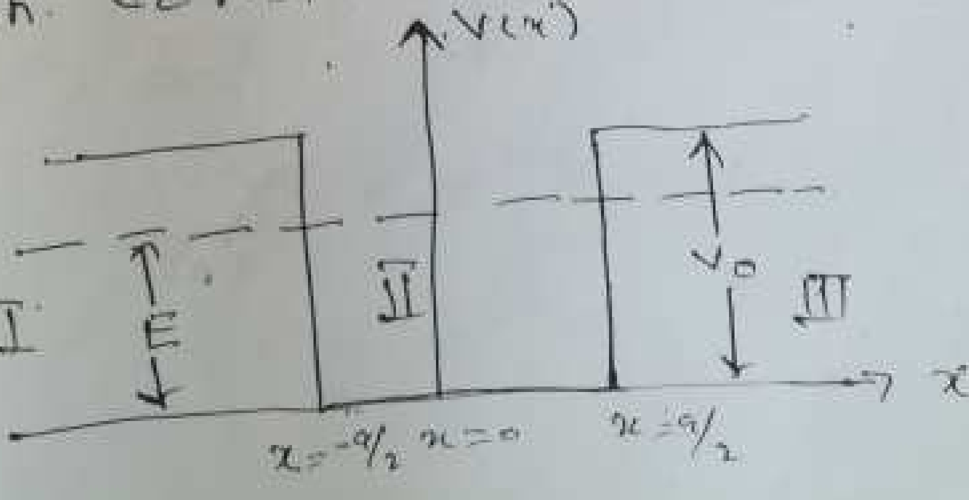
Consider a particle of mass "m" moving in following symmetric potential

$$V(x) = \begin{cases} V_0, & x < -a/2 \\ 0, & -a/2 \leq x \leq a/2 \\ V_0; & x > a/2 \end{cases}$$

The two physically interesting cases are $E > V_0$ and $E < V_0$. Here, we discuss only $E < V_0$.

The Bound State Solutions $(0 < E < V_0)$.

Classically, when $E < V_0$ the particle is completely confined to the region $-a/2 \leq x \leq a/2$, it will bounce back and forth b/w. $x = -a/2$ and $x = a/2$ with constant momentum $p = \sqrt{2mE}$



Quantum mechanically the solutions are interesting because they are supposed to be yield a discrete energy spectrum and wavefns that decay in the two regions $x < -a/2$ and $x > a/2$.

Hence, for each region the S.W.E can be written as

(I) $\underline{\underline{x < -a/2}}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + (V_0 - E)\psi_1 = 0$$

$$\frac{d^2\psi_1}{dx^2} - \frac{2m(V_0 - E)}{\hbar^2}\psi_1 = 0$$

$$\frac{d^2\psi_1}{dx^2} - K^2\psi_1 = 0, \quad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

(II) $\underline{\underline{-a/2 \leq x \leq a/2}}$

$$\frac{d^2\psi_2}{dx^2} + \frac{2mE}{\hbar^2}\psi_2 = 0$$

$$\frac{d^2\psi_2}{dx^2} + K_1^2\psi_2 = 0$$

where $K_1 = \frac{\sqrt{2mE}}{\hbar}$

(II) $x > -a/2$

$$\frac{d^2\psi_3}{dx^2} - \frac{2m(V_0-E)}{\hbar^2}\psi_3 = 0$$

$$\frac{d^2\psi_3}{dx^2} - k^2\psi_3 = 0$$

where $k = \frac{\sqrt{2m(V_0-E)}}{\hbar}$

The solutions are

(I). $\psi_1 = Ae^{kx} + Be^{-kx}$

Here $x \to -\infty$

so $e^{-kx} \to \infty$

so we set $B = 0$

then $\boxed{\psi_1 = Ae^{kx}}$

(II) $\boxed{\psi_2 = Ce^{ik_1x} + De^{-ik_1x}}$

(III) $\psi_3 = Ee^{kx} + Fe^{-kx}$

Here $x \to \infty$, then $e^{kx} \to \infty$

So $E = 0$

$$\boxed{\psi_3 = F e^{-Kx}}$$

Hence, our finite solutions are

(I) $\psi_1(x) = A e^{Kx}$ — $x < -a/2$

(II) $\psi_2(x) = C e^{iK_1 x} + D e^{-iK_1 x}$ — $-a/2 \leq x \leq a/2$

(III) $\psi_3(x) = F e^{-Kx}$ — $x > a/2$

Continuity ~~equa~~ condition gives.

(i) $\psi_1(-a/2) = \psi_2(-a/2)$

$$A e^{-Ka/2} = C e^{-iK_1 a/2} + D e^{iK_1 a/2} \quad \text{— (1)}$$

(ii)

$\frac{d}{dx}\psi_1(-a/2) = \frac{d}{dx}\psi_2(-a/2)$ gives

$$AK e^{-K\cancel{\phi}a/2} = iK_1 C e^{-iK_1 a/2} - iK_1 D e^{iK_1 a/2} \quad \text{— (2)}$$

(iii)

$\psi_2(a/2) = \psi_3(a/2)$

$$F e^{-Ka/2} = C e^{iK_1 a/2} + D e^{-iK_1 a/2} \quad \text{— (3)}$$

(iv) $\frac{d}{dx}\psi_2(a/2) = \frac{d}{dx}\psi_3(a/2)$

$$iK_1 C e^{iK_1 a/2} - iK_1 D e^{-iK_1 a/2} = \cancel{K} -K F e^{-Kx} \quad \text{— (4)}$$

By matching ① and ③, one will get (28 a)

$$(F+A)e^{-ka/2} = (C+D)\left[e^{ik_1a/2} + e^{-ik_1a/2}\right] \quad (5)$$

subtract ① from ③

$$(F-A)e^{-ka/2} = (C-D)\left[e^{ik_1a/2} - e^{-ik_1a/2}\right] \quad (6)$$

Now add ② and ④

$$K(A-F)e^{-ka/2} = ik_1(C-D)\left[e^{ik_1a/2} + e^{-ik_1a/2}\right] \quad (7)$$

subtracting ④ from ②

$$K(A+F)e^{-ka/2} = ik_1(C+D)\left[e^{-ik_1a/2} - e^{ik_1a/2}\right] \quad (8)$$

multiply ⑥ by K both side and take -ve sign common from L.H.S of ⑥ and add with ⑦, one gets

$$-K = ik_1\left[\frac{e^{ik_1a/2} + e^{-ik_1a/2}}{e^{ik_1a/2} - e^{-ik_1a/2}}\right]$$

$$-K = ik_1\left[\frac{\cos k_1a/2 + i\cancel{\sin k_1a/2} + \cos k_1a/2 - i\cancel{\sin k_1a/2}}{\cancel{\cos k_1a/2} + i\sin k_1a/2 - \cancel{\cos k_1a/2} + i\sin k_1a/2}\right]$$

$$-K = 2K_1\left[\frac{2\cos k_1 a/2}{2i\sin k_1 a/2}\right]$$

$$-K = k_1 \cot k_1 a/2$$

$$\boxed{K = -k_1 \cot(k_1 a/2)} \leftarrow (*)$$

This is for odd parity states (anti-symmetric)

Similarly by simplyfing (5) and (8), one gets

$$\boxed{K = k_1 \tan(k_1 a/2)} \leftarrow (**)$$

for even parity states. – (symmetric)

The two equation $(*)$ and $(**)$ can not be solved simultaneously, therefore, we look at the qualitative solution of the eigen-value problem, as

let $\xi = Ka/2$

Putting the value of $K = \frac{\sqrt{2m(V_0-E)}}{\hbar}$

$$\xi = \frac{\sqrt{2m(V_0-E)}}{2\hbar}\, a$$

$$\zeta^2 = \frac{2m(V_0 - E)}{4\hbar^2} a^2 \quad \text{— (A)}$$

Now $$\eta = \frac{k_1 a}{2}$$

putting the value of $k_1 = \frac{\sqrt{2mE}}{\hbar}$

$$\eta = \frac{\sqrt{2mE}}{2\hbar} a$$

squaring on both side

$$\eta^2 = \frac{2mE}{4\hbar^2} a^2 \quad \text{— (B)}$$

By adding equ (A) and (B)

$$\zeta^2 + \eta^2 = \frac{a^2 2mV_0 - 2mEa^2}{4\hbar^2} + \frac{2mEa^2}{4\hbar^2}$$

$$\boxed{\zeta^2 + \eta^2 = \frac{mV_0 a^2}{\hbar^2} = r^2}$$

This represents the circle with radius r.

The eqn(*) and (**) take the form (30)

$$\xi = \eta \tan \eta$$

and

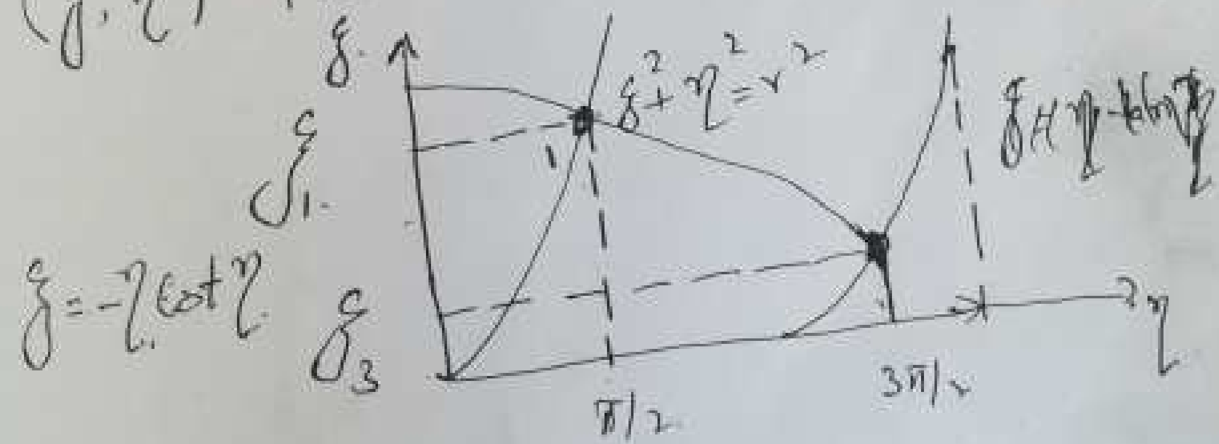
$$\xi = -\eta \cot \eta$$

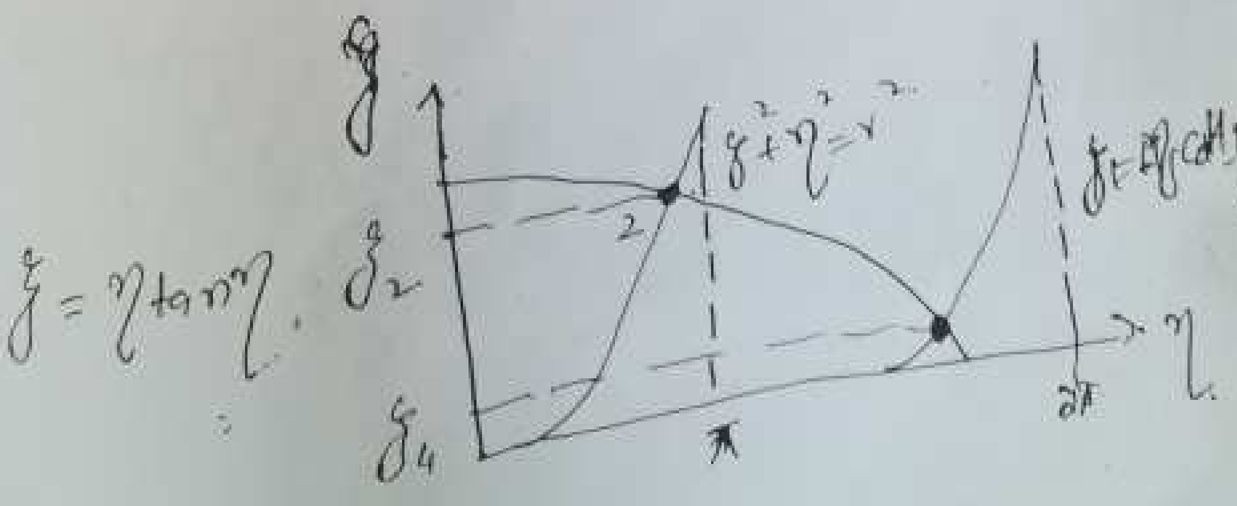
The desired energy values can be obtained by constructing the intersection of two curves

$$\xi = \eta \tan \eta$$

$$\xi = -\eta \cot \eta$$

with the circle $\xi^2 + \eta^2 = r^2$, within (ξ, η) plane.





The intersection of the curves $\xi = \eta \tan \eta$ and $\xi = -\eta \cot \eta$ with the circle $\xi^2 + \eta^2 = r^2$ determine the energy eigenvalues of the problem. The number of energy levels increases with V_0 and a.

For $V_0 a^2 \longrightarrow \infty$, the intersection take place at

$$\tan \eta = \tan\left(\frac{k_1 a}{2}\right) = \infty$$

This corresponds to the values

$$\frac{k_1 a}{2} = \pi/2,\ 3\pi/2,\ 5\pi/2$$

$$\frac{k_1 a}{2} = \frac{(2n+1)}{2}\pi, \quad n = 0, 1, 2, \ldots$$

$$k_1 a = (2n+1)\pi \text{ — } \cancel{\text{odd}}\ \text{even } n = 1, 3$$

Similarly

$$-\cot\left(\frac{k_1 a}{2}\right) = 0 \text{ — corresponds to}$$

$$\frac{k_1 a}{2} = \frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{\ }\ldots$$

$$\frac{k_1 a}{\cancel{\pi}} = \cancel{2}n\pi \text{ — } \quad n = 1, 2, 3, \ldots$$

Combining these two results

$$K_1 a = n\pi$$

The energy eigenvalues are obtained as

$$\eta = \frac{K_1 a}{2} = \frac{\sqrt{2mE_n}}{\hbar} a/2$$

$$(K_1 a)^2 = \frac{2mE_n}{\hbar^2} a^2$$

$$(n\pi)^2 = \frac{2mE_n}{\hbar^2} a^2$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$