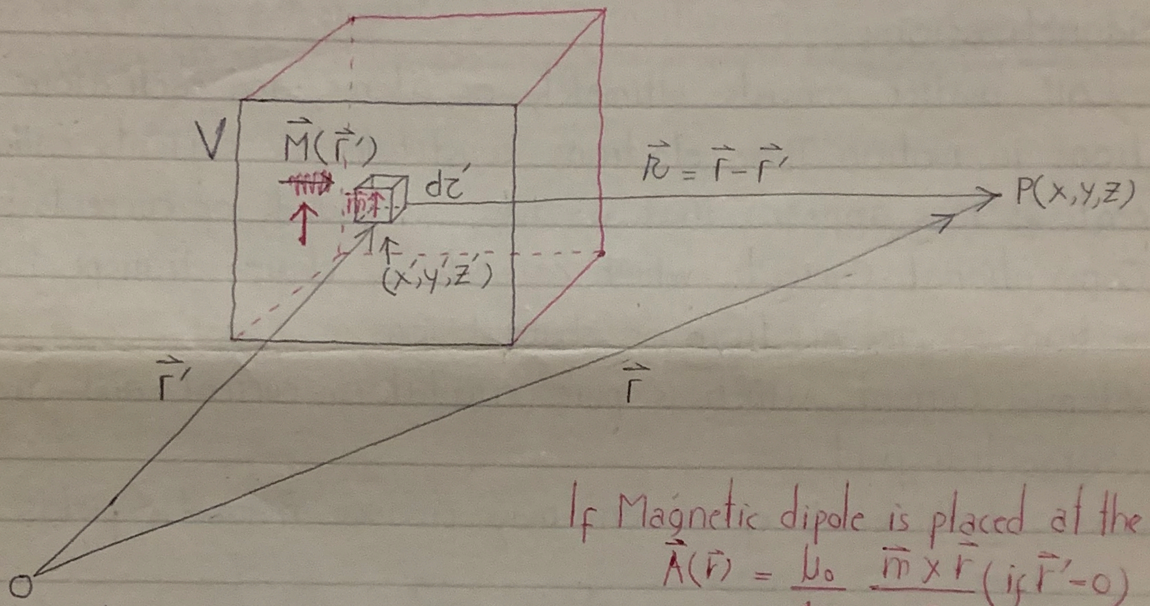
The Field of a Magnetized Object: (Bound Currents)



If Magnetic dipole is placed at the origin

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad (\text{if } \vec{r}' = 0)$$

For a magnetic dipole, we have

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \vec{A}(\vec{r}) \; ; \; \vec{m} = \vec{m}(\vec{r}')?$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

electric potential of an electric dipole.

In the present context, ?

$$d\vec{m} = \vec{M}(\vec{r}')\, d\tau'$$

$$d\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{M}(\vec{r}')\, d\tau' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

The vector potential at point 'P' due to the whole block of magnetized material is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}\, d\tau'$$

As, $$\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \vec{M}(\vec{r}') \times \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) d\tau'$$

Using vector identity,

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

$$\vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = \frac{1}{|\vec{r}-\vec{r}'|}(\vec{\nabla}'\times\vec{M}(\vec{r}')) - \vec{M}(\vec{r}')\times\left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\Rightarrow \vec{M}\times\left(\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right) = \frac{1}{|\vec{r}-\vec{r}'|}(\vec{\nabla}'\times\vec{M}) - \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{1}{|\vec{r}-\vec{r}'|}(\vec{\nabla}'\times\vec{M}(\vec{r}'))\, d\tau' - \frac{\mu_0}{4\pi}\int_V \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) d\tau'$$

Using vector identity,

$$\underbrace{\int_V (\vec{\nabla}\times\vec{v})\, d\tau = -\oint_S \vec{v}\times d\vec{a}}_{\text{Assignment for the students}}$$ Problem 1.60 (b) in David J. Griffiths.

Hints: Use Divergence theorem, $\int(\vec{\nabla}\cdot\vec{v})\, d\tau = \oint \vec{v}\cdot d\vec{a}$

replace $\vec{v}$ by $\vec{v}\times\vec{c}$; $\vec{c}$ = constant vector

Let, $\vec{v} \to \vec{v}\times\vec{c}$; use it in Divergence theorem, we get

$$\int_V [\vec{\nabla}\cdot(\vec{v}\times\vec{c})]\, d\tau = \oint_S (\vec{v}\times\vec{c})\cdot d\vec{a}$$

Using vector identity

$$\int_V [\vec{c}\cdot(\vec{\nabla}\times\vec{v}) - \vec{v}\cdot(\underbrace{\vec{\nabla}\times\vec{c}}_{=0})]\, d\tau = \oint_S (\vec{v}\times\vec{c})\cdot d\vec{a}$$

0 (since, $\vec{c}$ is any constant vector)

$$\int_V \vec{c}\cdot(\vec{\nabla}\times\vec{v})\, d\tau = -\oint_S (\vec{c}\times\vec{v})\cdot d\vec{a}$$

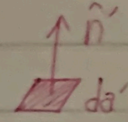
$$= -\oint \vec{c}\cdot(\vec{v}\times d\vec{a})$$

$$\vec{c}\cdot\int_V(\vec{\nabla}\times\vec{v})\, d\tau = \vec{c}\cdot\left[-\oint_S \vec{v}\times d\vec{a}\right]$$

$$\Rightarrow \int_V(\vec{\nabla}\times\vec{v})\, d\tau = -\oint_S \vec{v}\times d\vec{a}$$; Hence, proved.

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{(\vec{\nabla}' \times \vec{M})}{|\vec{r}-\vec{r}'|} d\tau' + \frac{\mu_0}{4\pi}\oint \frac{\vec{M}}{|\vec{r}-\vec{r}'|} \times d\vec{a}'$$

$$= \frac{\mu_0}{4\pi}\int \frac{[\vec{\nabla}' \times \vec{M}(\vec{r}')]}{|\vec{r}-\vec{r}'|} d\tau' + \frac{\mu_0}{4\pi}\oint \frac{\vec{M}(\vec{r}') \times \hat{n}'}{|\vec{r}-\vec{r}'|} da'$$



where, $\hat{n}' \equiv$ unit vector normal to the area element da'

The first term looks like the magnetic vector potential of a volume current

SI Unit: $\frac{1}{m} \cdot \frac{A}{m} = A/m^2$ $\quad \vec{J}_b(\vec{r}') = \vec{\nabla}' \times \vec{M}(\vec{r}') \equiv$ bound volume current density

The second term looks like the potential of a surface current

SI Unit: A/m $\quad \vec{K}_b(\vec{r}') = \vec{M}(\vec{r}') \times \hat{n}' \equiv$ bound surface current density

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_b(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau' + \frac{\mu_0}{4\pi}\oint \frac{\vec{K}_b(\vec{r}')}{|\vec{r}-\vec{r}'|} da'$$

Notice the striking parallel with the electrostatic case:

$\rho_b(\vec{r}') = -\vec{\nabla}' \cdot \vec{P}(\vec{r}') \equiv$ bound volume charge density

$\sigma_b(\vec{r}') = \vec{P}(\vec{r}') \cdot \hat{n}' \equiv$ bound surface charge density

Now, we can find $\vec{B}$ by $\vec{B} = \vec{\nabla} \times \vec{A}$, but as the curl operation is to be performed twice which is difficult to evaluate. Thus, we will consider an alternate method to find $\vec{B}$.