Kinematics

The branch of mechanics which deals with the motion of objects is known as dynamics.

Dynamics

Kinetics — Kinematics

$F = ma$

Kinetics: (Discusses forces under which motion takes place)

Kinematics: (Describes the motion of the body without any reference to the forces acting on it)

Def: A particle is a point body, i.e., a body which has no dimensions.

VELOCITY AND ACCELERATION

Let $\vec{r}$ be position vector of a particle. As the particle moves, the vector $\vec{r}$ changes with time t.

The curve traced by the particle as the time progresses is called the <u>Trajectory</u> or <u>path of the particle</u>.

The path of the particle is specified by

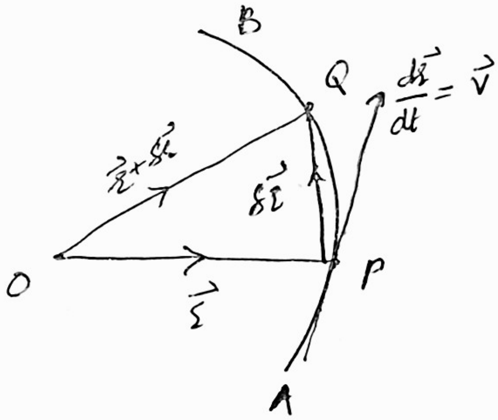
$$\vec{r} = \vec{r}(t) \longrightarrow (1)$$

In component form (1) can be written as

$$x = x(t), \quad y = y(t), \quad z = z(t) \longrightarrow (2)$$

Eqs. (2) are parametric eqs. of path.

Let AB be a part of trajectory of the particle as shown in fig.



Let particle be at point P at time t whose p.v. is $\vec{r}$. After a small time δt, particle reach the point Q whose p.v. is $\vec{r}+\delta\vec{r}$.

Then $\quad \vec{PQ} = \delta\vec{r}$

then $\frac{\delta\vec{r}}{\delta t}$ = Average rate of displacement of the particle in the interval δt.

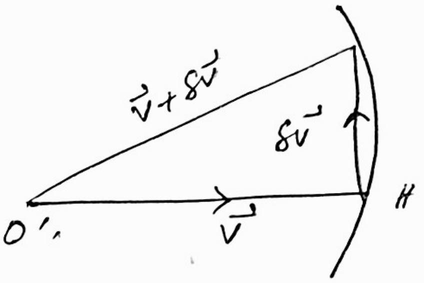
In the limiting case when $\delta t \to 0$, $\delta\vec{r} \to 0$ and $Q \to P$, then

$\lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} =$ Instantaneous rate of displacement

This is defined as the instantaneous velocity or simply velocity $\vec{v}$ of the particle at point P.

$$\boxed{\lim_{\delta t \to 0} \frac{\delta \vec{s}}{\delta t} = \frac{d\vec{s}}{dt} = \vec{v}}$$

Let O′ be a fixed point. Draw a vector $\vec{O'H}$ such that it represents the velocity of the particle at any time t.



Point H moves continuously if the magnitude of the velocity or direction of velocity or both change with time. The curve described by H is called hodograph of the motion of the particle. We say that

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \frac{d\vec{v}}{dt}$$

$\vec{a}$ is along the tangent to the hodograph. We can write

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{s}}{dt}\right) = \frac{d^2\vec{s}}{dt^2}$$

Example A particle is moving in such a way that its position at any time t is specified by

$$\vec{r} = (t^3+t^2)\hat{i} + (\cos t + \sin^2 t)\hat{j} + (e^t + \log t)\hat{k}$$

Find its velocity and acceleration.

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = (3t^2+2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + (e^t + \frac{1}{t})\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (6t+2)\hat{i} + (2\cos 2t - \cos t)\hat{j} + (e^t - \frac{1}{t^2})\hat{k}$$

CARTESIAN COMPONENTS OF VELOCITY AND ACCELERATION

In Cartesian coordinates, we have

$$\vec{r} = x\hat{i} + y\hat{j}$$

So $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

So velocity $\vec{v}$ has components $\frac{dx}{dt}$ and $\frac{dy}{dt}$ along the coordinate axes.

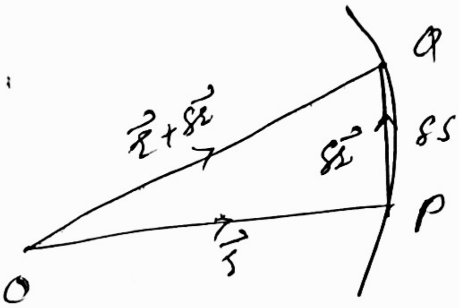
Also $$v = |\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

where s is the distance of the particle along the path from some fixed point on the path.

Now $\vec{v} = \dfrac{d\vec{r}}{dt}$

$= \dfrac{d\vec{r}}{ds} \cdot \dfrac{ds}{dt}$

$= v \dfrac{d\vec{r}}{ds}$



To find magnitude and direction of $\dfrac{d\vec{r}}{ds}$

Since $\dfrac{d\vec{r}}{ds} = \lim_{\delta s \to 0} \dfrac{\delta\vec{r}}{\delta s}$,

this vector is ~~tangent~~ parallel to the tangent to the path at P and its magnitude is

$$\left| \lim_{\delta s \to 0} \frac{\delta \vec{r}}{\delta s} \right| = \lim_{\delta s \to 0} \frac{|\delta \vec{r}|}{\delta s} = 1$$

Because limit of the ratio of the length of the chord PQ to the length of the arc PQ is unity.

thus $\dfrac{d\vec{r}}{ds} = \hat{t}$

where $\hat{t}$ is a unit vector parallel to the tangent at P. The vector $\hat{t}$ is called unit tangent at P.

Thus $\vec{v} = v\hat{t}$

This eq. shows that at any instant the particle is moving in the direction of the tangent to the path.

Now $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

Showing that $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ are components of acceleration along coordinate axes.

Example. At any time t, the position of a particle moving in a plane, can be specified by $(a \cos wt, a \sin wt)$, where a and w are constants. Find the components of its velocity and acceleration along the coordinate axes.

Sol. For finding the required components along the x-axis, we notice that

$$x = a \cos wt.$$

Hence the component of its velocity along the x-axis is

$$\frac{dx}{dt} = -a\, w \sin wt$$

and the component of its acceleration along that line is

$$\frac{d^2x}{dt^2} = -a\, w^2 \cos wt.$$

Similarly the components of velocity and acceleration along the y-axis are $a\, w \cos wt$ and $-a\, w^2 \sin wt$ respectively.

7.4 Tangential and Normal Components of Velocity and Acceleration

We have seen that the velocity of a particle is entirely along the tangent to the path (*see* equation (7.9) and, therefore, has no component along the normal to the path. Let us now find the tangential and the normal components of acceleration. This can be done by differentiating equation (7.9) with respect to t. We get

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(v\,\mathbf{t})=\frac{dv}{dt}\mathbf{t}+v\frac{d\mathbf{t}}{dt}. \tag{7.11}$$

Now $\mathbf{t}$ is a unit vector.

$$\therefore \quad \mathbf{t}.\mathbf{t}=1. \tag{7.12}$$

Differentiating equation (7.12) with respect to t, we get

$$\frac{d\mathbf{t}}{dt}.\mathbf{t}+\mathbf{t}.\frac{d\mathbf{t}}{dt}=0$$

or

$$\mathbf{t}.\frac{d\mathbf{t}}{dt}=0, \tag{7.13}$$

which shows that either the vector $\frac{d\mathbf{t}}{dt}$ vanishes or it is perpendicular to $\mathbf{t}$. Since, in general, $\frac{d\mathbf{t}}{dt}$ cannot be zero as the direction of $\mathbf{t}$ is usually changing, we conclude that $\mathbf{t}$ and $\frac{d\mathbf{t}}{dt}$ are mutually perpendicular. Let the unit vector in the direction of $\frac{d\mathbf{t}}{dt}$ be $\mathbf{n}$. Then

$$\mathbf{n}.\mathbf{t}=0, \tag{7.14}$$

because $\mathbf{n}$ and $\mathbf{t}$ are orthogonal.

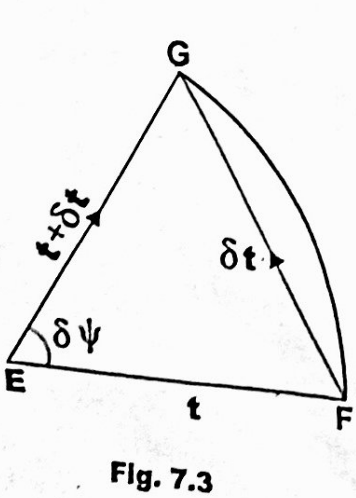


Fig. 7.3

Let us now find the magnitude of $\frac{d\mathbf{t}}{dt}$. We can write

$$\left|\frac{d\mathbf{t}}{dt}\right|=\underset{\delta t\to 0}{Lt}\left|\frac{\delta\mathbf{t}}{\delta t}\right|=\underset{\substack{\delta t\to 0\\ \delta\psi\to 0\\ \delta s\to 0}}{Lt}\left|\frac{\delta\mathbf{t}}{\delta\psi}\cdot\frac{\delta\psi}{\delta s}\cdot\frac{\delta s}{\delta t}\right|, \tag{7.15}$$

where $\delta\psi$ is the angle through which the unit tangent turns in time δt (*see* Fig. 7.3).

Now the length of the arc described by the point F in going to G is $1.\delta\psi=\delta\psi$ and the length of the chord FG is $|\delta\mathbf{t}|$ so that $\underset{\delta\psi\to 0}{Lt}\left|\frac{\delta\mathbf{t}}{\delta\psi}\right|=1$. Then

$$\left|\frac{d\mathbf{t}}{dt}\right|=\frac{ds}{dt}\frac{d\psi}{ds}=v\frac{d\psi}{ds}, \tag{7.16}$$

where $\frac{d\psi}{ds}$ is the arc-rate at which tangent to the path turns, i.e., it is the curvature of the path and is non-negative.

Thus
$$\frac{d\mathbf{t}}{dt}=\left|\frac{d\mathbf{t}}{dt}\right|\mathbf{n}=\kappa v\,\mathbf{n}=\frac{v}{\rho}\mathbf{n}, \tag{7.17}$$

where κ is the curvature and $\rho=\frac{1}{\kappa}$, is the radius of curvature of the path of the particle. Equation (7.11) can now be written as

$$\mathbf{a}=\frac{dv}{dt}\mathbf{t}+\frac{v^2}{\rho}\mathbf{n}. \tag{7.18}$$

Equation (7.18) shows that the tangential component of the acceleration is $\frac{dv}{dt}$ and the normal component is $\frac{v^2}{\rho}$. It is clear that if the speed v remains unchanged with time, then $\frac{dv}{dt}=0$ and the acceleration of the particle is $\frac{v^2}{\rho}$ and is wholly in the direction of the normal to the path. This is not surprising because this acceleration is a consequence of the change in the direction of $\mathbf{v}$ along the path.

For a circle the radius of curvature is equal to its radius so that a particle moving along a circle of radius r with velocity $\mathbf{v}$ whose magnitude v is constant, is subject to an acceleration $\frac{v^2}{\rho}$ directed towards the centre of the circle.

For a straight line the radius of curvature is infinite so that a particle moving along a straight line with a constant velocity, does not have any acceleration.

Example. A particle is moving along the parabola $x^2=4ay$ with constant speed v. Determine the tangential and the normal components of its acceleration when it reaches the point whose abscissa is $\sqrt{5}\,a$.

Sol. The tangential component $\frac{dv}{dt}$ is zero because v is constant. The normal component is given by $\frac{v^2}{\rho}$, where ρ is the radius of curvature at the point under consideration. From Differential Calculus we know that

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

For the parabola $x^2=4\,ay$, we have

$$\frac{dy}{dx}=\frac{x}{2a} \text{ and } \frac{d^2y}{dx^2}=\frac{1}{2a}.$$

The radius of curvature of the parabola at a point whose abscissa is x, is given by

$$\rho=2a\left[1+\left(\frac{x}{2a}\right)^2\right]^{\frac{3}{2}}=\frac{(4a^2+x^2)^{\frac{3}{2}}}{(2a)^2}.$$

For $x=\sqrt{5}\,a$, we get $\rho=\frac{27}{4}a$.

Hence the normal component of the acceleration is $\frac{4\,v^2}{27\,a}$.

7.5 Radial and Transverse Components of Velocity and Acceleration

In plane polar coordinates, the position of a particle is specified by a radius vector r and the polar angle θ which are related to x and y through the relations

$$x=r\cos\theta,$$
$$y=r\sin\theta,$$

provided the two coordinate frames have the same origin and the x-axis and the initial line coincide. The direction of the radius vector is known as the *radial direction* and that perpendicular to it in the direction of increasing θ is called the *transverse direction*.

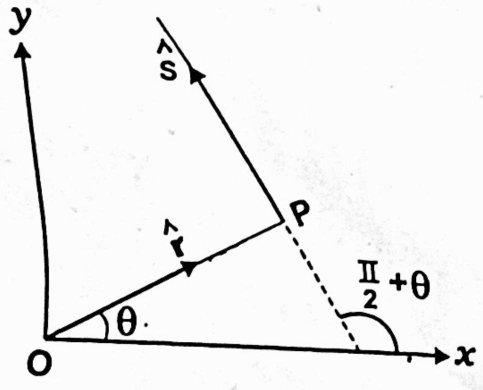


Fig. 7.4

Let $\hat{\mathbf{r}}, \hat{\mathbf{s}}$ be unit vectors in the radial and transverse directions respectively as shown in Fig. 7.4. Then

$$\hat{\mathbf{r}}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j},$$

$$\hat{\mathbf{s}}=\cos\left(\frac{\pi}{2}+\theta\right)\mathbf{i}+\sin\left(\frac{\pi}{2}+\theta\right)\mathbf{j}$$

$$=-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j},$$

so that
$$\frac{d\hat{\mathbf{r}}}{dt}=(-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j})\frac{d\theta}{dt}$$
$$=\frac{d\theta}{dt}\hat{\mathbf{s}},$$
$$\frac{d\hat{\mathbf{s}}}{dt}=(-\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j})\frac{d\theta}{dt}$$
$$=-\frac{d\theta}{dt}\hat{\mathbf{r}}.$$

Now
$$\mathbf{v}=\frac{d\mathbf{r}}{dt}$$
$$=\frac{d}{dt}(r\hat{\mathbf{r}})$$
$$=\frac{dr}{dt}\hat{\mathbf{r}}+r\frac{d\hat{\mathbf{r}}}{dt}$$
$$=\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\mathbf{s}}, \tag{7.19}$$
where dot denotes differentiation with respect to time t.

Thus radial and transverse components v_r, v_θ of the velocity are $\dot{r}$, $r\dot{\theta}$ respectively.

Also
$$\mathbf{a}=\frac{d\mathbf{v}}{dt}$$
$$=\frac{d}{dt}(\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\mathbf{s}})$$
$$=\ddot{r}\hat{\mathbf{r}}+\dot{r}\dot{\theta}\hat{\mathbf{s}}+\dot{r}\dot{\theta}\hat{\mathbf{s}}+r\ddot{\theta}\hat{\mathbf{s}}+r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}})$$
$$=\{\ddot{r}-r(\dot{\theta})^2\}\hat{\mathbf{r}}+(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\mathbf{s}}. \tag{7.20}$$

Therefore, the radial and transverse components of the acceleration are
$$a_r=\ddot{r}-r(\dot{\theta})^2, \quad a_\theta=2\dot{r}\dot{\theta}+r\ddot{\theta}.$$
The latter component is usually written as
$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

Cor. If a particle describes a circle of radius a, $\dot{r}=0$, and the transverse direction is that of the tangent to the circle. In such a case, the velocity $a\dot{\theta}$ is entirely along the tangent and acceleration components are: $a(\dot{\theta})^2$ along the inward radius, $a\ddot{\theta}$ along the tangent.

Example. A particle P moves in a plane in such a way that at any time t, its distance from a fixed point O is $r=at+bt^2$ and the line connecting O and P makes an angle $\theta=ct^{\frac{3}{2}}$ with a fixed line OA. Find the radial and transverse components of the velocity and acceleration of the particle at $t=1$.

Sol. In this case

$$r=at+bt^2, \qquad \theta=ct^{\frac{3}{2}}$$

so that
$$\dot{r}=a+2bt, \qquad \dot{\theta}=\tfrac{3}{2}\,ct^{\frac{1}{2}}.$$

Hence the radial velocity $v_r=a+2bt$,

and the transverse velocity $v_\theta=\frac{3}{2}\,ct^{\frac{1}{2}}\,(at+bt^2)$.

At $t=1$,

$$v_r=a+2b,$$
$$v_\theta=\tfrac{3}{2}\,c(a+b).$$

Again $\ddot{r}=2b, \quad \ddot{\theta}=\frac{3}{4}\,c\,\frac{1}{\sqrt{t}}.$

At $t=1$,

$$r=a+b, \qquad \theta=c$$
$$\dot{r}=a+2b, \quad \dot{\theta}=\tfrac{3}{2}\,c$$
$$\ddot{r}=2b, \qquad \ddot{\theta}=\tfrac{3}{4}\,c.$$

Hence the required components are given by

$$a_r=\ddot{r}-r(\dot{\theta})^2=2b-(\tfrac{3}{2}\,c)^2\,(a+b)=\tfrac{1}{4}\,[8b-9c^2\,(a+b)]$$

and
$$a_\theta=r\,\ddot{\theta}+2\dot{r}\,\dot{\theta}=(a+b)\tfrac{3}{4}\,c+2.\ (a+2b)\tfrac{3}{2}\,c.$$
$$=\tfrac{3}{4}\,c\,(a+b+4a+8b)$$
$$=\tfrac{3}{4}\,c\,(5a+9b).$$

Exercises Set 7

1. A particle starts from O at $t=0$. Find its velocity and acceleration at any time t if its position at that time is given by

   (i) $\mathbf{r}=(t^3+2t)\mathbf{i}+(5t^2-7)\,\mathbf{j}$,

   (ii) $\mathbf{r}=at^2\mathbf{i}+4at\,\mathbf{j}$,