$\therefore$ the position of the *c.m.* of the whole lamina is given by

$$\bar{x}=\frac{\int_0^a \pi x\lambda\, x^2 \frac{2x}{\pi}\, dx}{\int_0^a \pi x \lambda x^2\, dx}$$

$$=\frac{2}{\pi}\frac{\int_0^a x^4\, dx}{\int_0^a x^3\, dx}=\frac{2}{\pi}\frac{\frac{a^5}{5}}{\frac{a^4}{4}}$$

$$=\frac{8a}{5\pi}.$$

4.7 Centre of Mass of an Arc

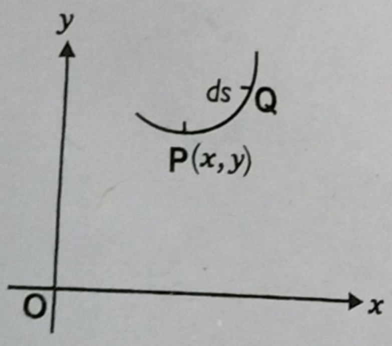

Fig. 4.19

We have found the *c.m.* of arcs in some special cases from considerations of symmetry. We now derive general formulae for the coordinates of the *c.m.* of any arc of a given curve shown in Fig. 4·19.

Let ρ be the density of a plane curve at the point (x, y) and ds the arc element at the same point. Then an element of mass is given by

$$dm=\rho\, ds$$

$\therefore$ formulae for the *c.m.* become

$$\bar{x}=\frac{\int \rho x\, ds}{\int \rho\, ds},\quad \bar{y}=\frac{\int \rho y\, ds}{\int \rho\, ds}. \tag{4.19}$$

In case of a uniform arc the *c.m.* (which now coincides with the centroid of the arc) is given by

$$\bar{x}=\frac{\int x\, ds}{\int ds},\quad \bar{y}=\frac{\int y\, ds}{\int ds}. \tag{4.20}$$

N.B. In terms of Cartesian coordinates

$$ds^2=dx^2+dy^2 \quad \text{or} \quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

and in terms of polar coordinates

$$ds^2=dr^2+r^2\, d\theta^2 \quad \text{or} \quad ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

If x, y are given in terms of a parameter t, we may use the result

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$$

Example. Find the centroid of the arc of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ lying in the first quadrant.

Sol. ***First Method :***

The equation of the curve gives

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0.$$

$$\therefore\quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

$$\therefore\quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}\,dx$$

$$=\frac{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{x^{\frac{1}{3}}}\,dx$$

$$=\left(\frac{a}{x}\right)^{\frac{1}{3}}dx. \quad [\because\ (x,y) \text{ lies on the given curve}]$$

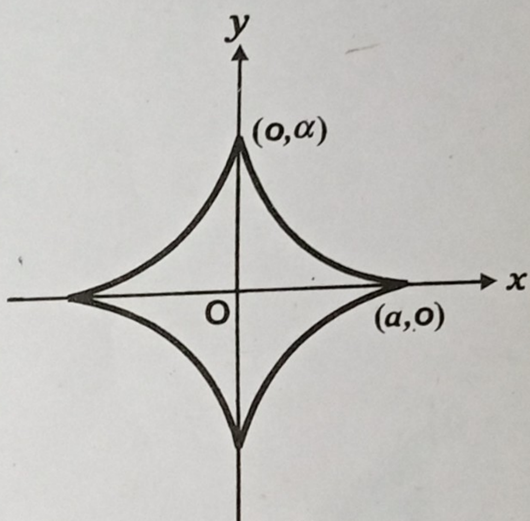

Fig. 4.20

$$\therefore\quad \bar{x}=\frac{\int_0^a x\left(\frac{a}{x}\right)^{\frac{1}{3}}dx}{\int_0^a\left(\frac{a}{x}\right)^{\frac{1}{3}}dx}=\frac{\int_0^a x^{\frac{2}{3}}\,dx}{\int_0^a x^{-\frac{1}{3}}\,dx}$$

$$=\frac{\left|\frac{3}{5}x^{\frac{5}{3}}\right|_0^a}{\left|\frac{3}{2}x^{\frac{2}{3}}\right|_0^a}=\frac{2}{5}a.$$

By symmetry $\bar{y}=\bar{x}=\frac{2}{5}a$.

Second Method :

The given curve has the parametric equations

$$x = a\cos^3\theta,\ y = a\sin^3\theta.$$

$$\therefore\ \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

$$\therefore\ ds = \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ d\theta$$

$$= 3a\sin\theta\cos\theta\ d\theta.$$

Also, as x varies from 0 to a, θ varies from $\frac{\pi}{2}$ to 0.

$$\therefore\ \bar{x} = \frac{\int_{\pi/2}^{0} a\cos^3\theta .\ 3a\sin\theta\cos\theta\ d\theta}{\int_{\pi/2}^{0} 3a\sin\theta\cos\theta\ d\theta}$$

$$= a\frac{\int_0^{\pi/2}\cos^4\theta\sin\theta\ d\theta}{\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta} = a\frac{\dfrac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\,\Gamma\left(\frac{4+1}{2}+1\right)}}{\dfrac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\,\Gamma\left(\frac{1+1}{2}+1\right)}}$$

$$= a\frac{\dfrac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(\frac{7}{2})}}{\dfrac{\Gamma(1)\Gamma(1)}{\Gamma(2)}}$$

$$= \frac{\dfrac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}}{\dfrac{1}{1}}\,a = \frac{2}{5}a = \bar{y}\ \text{(By symmetry)}.$$

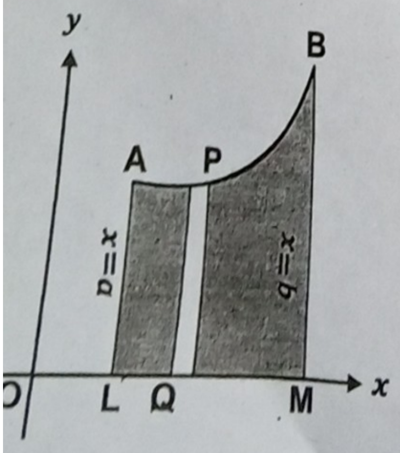

Fig. 4.21

4.8 Centroid of a Plane Region

(1) Consider the region bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$ as shown in Fig. 4·21. The area of an elementary strip parallel to y-axis at distance x from it is $y\,\delta x$.

The coordinates of the centroid of the strip are $\left(x, \frac{y}{2}\right)$.

∴ coordinates of the centroid of the given region are

$$\bar{x}=\frac{\int_a^b xy\,dx}{\int_a^b y\,dx},$$

$$\bar{y}=\frac{\int_a^b \frac{y}{2}\cdot y\,dx}{\int_a^b y\,dx}=\frac{1}{2}\frac{\int_a^b y^2\,dx}{\int_a^b y\,dx}.$$

(2) Formulae for the centroid of the region bounded by the curve $r=f(\theta)$ and the radius vectors $\theta=\theta_1$, $\theta=\theta_2$ may be obtained as follows :

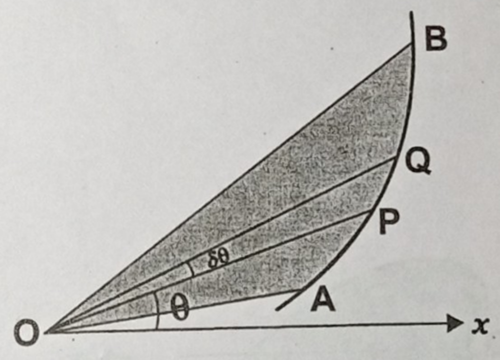

Fig. 4.22

Let $\widehat{AB}$ be the curve $r=f(\theta)$ and OA, OB the radius vectors $\theta=\theta_1$, $\theta=\theta_2$ as shown in Fig. 4·22. Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points of the curve.

Then the area of the sectorial element POQ

$$=\tfrac{1}{2}r(r+\delta r)\sin\delta\theta$$

$$=\tfrac{1}{2}r(r+\delta r)\,\delta\theta,$$

(Since $\delta\theta$ in the limit tends to zero.

$$=\tfrac{1}{2}r^2\,\delta\theta.$$

(To first order small quantities.)

Also the coordinates of the centroid of the elementary area are $(\tfrac{2}{3}r\cos\theta, \tfrac{2}{3}r\sin\theta)$. Hence the centroid of the total area is given by

$$\bar{x}=\frac{\int_{\theta_1}^{\theta_2}\tfrac{2}{3}r\cos\theta\cdot\tfrac{1}{2}r^2 d\theta}{\int_{\theta_1}^{\theta_2}\tfrac{1}{2}r^2 d\theta}=\frac{\tfrac{2}{3}\int_{\theta_1}^{\theta_2}r^3\cos\theta\,d\theta}{\int_{\theta_1}^{\theta_2}r^2 d\theta},$$

$$\bar{y}=\frac{\int_{\theta_1}^{\theta_2}\tfrac{2}{3}r\sin\theta\cdot\tfrac{1}{2}r^2 d\theta}{\int_{\theta_1}^{\theta_2}\tfrac{1}{2}r^2 d\theta}=\frac{\tfrac{2}{3}\int_{\theta_1}^{\theta_2}r^3\sin\theta\,d\theta}{\int_{\theta_1}^{\theta_2}r^2 d\theta}.$$

Example 1. Find the centroid of the region bounded by the coordinate axes and the circle $x^2+y^2=a^2$ which lies in the first quadrant.

Sol.
$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx}=\frac{\int_0^a x\sqrt{a^2-x^2}\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx}$$

$$=\frac{\int_0^a \frac{1}{2}.\,2x\sqrt{a^2-x^2}\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx}=\frac{\left|-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right|_0^a}{\left|\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a}$$

$$=\frac{\frac{a^3}{3}}{a^2\frac{\pi}{4}}=\frac{4a}{3\pi}=\bar{y}\quad\text{(By symmetry).}$$

Example 2. Find the distance from the cusp of the centroid of the region bounded by the cardioide

$$r=a(1+\cos\theta).$$

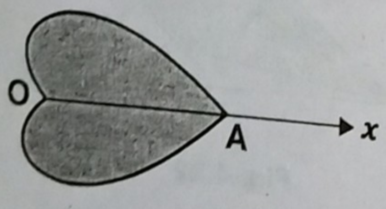

Fig. 4.23

Sol. By symmetry the centroid lies on the initial line. Also the x-coordinate of the centroid of the upper half is the same as that of the whole region (*see* Fig. 4·23).

∴ taking limits of θ for the upper half only,

$$\bar{x}=\frac{\frac{2}{3}\int_0^\pi r^3\cos\theta\,d\theta}{\int_0^\pi r^2 d\theta}$$

$$=\frac{\frac{2}{3}\int_0^\pi a^3(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^\pi a^2(1+\cos\theta)^2 d\theta}$$

$$=\frac{2}{3}a.\frac{\int_0^\pi(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta}{\int_0^\pi(1+2\cos\theta+\cos^2\theta)\,d\theta}.$$

The integral in the numerator is

$$I_1=\int_0^{\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta.$$

Now $\cos(\pi-\theta)=-\cos\theta$ and $\cos^3(\pi-\theta)=-\cos^3\theta$.

But $\cos^2(\pi-\theta)=\cos^2\theta$, $\cos^4(\pi-\theta)=\cos^4\theta$.

$$\therefore\quad I_1=2\int_0^{\pi/2}(3\cos^2\theta+\cos^4\theta)\,d\theta.$$

$$=2\left(3.\frac{1}{2}\cdot\frac{\pi}{2}+\frac{3}{4}.\frac{1}{2}\cdot\frac{\pi}{2}\right)=2.\frac{15}{16}\pi$$

$$=\frac{15}{8}\pi.$$

Similarly, the integral in the denominator

$$I_2=\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=2\int_0^{\pi/2}(1+\cos^2\theta)\,d\theta$$

$$=2.\left(\frac{\pi}{2}+\frac{1}{2}.\frac{\pi}{2}\right)=\frac{3}{2}\pi.$$

$$\therefore\quad \bar{x}=\frac{2}{3}a\times\frac{\frac{15}{8}\pi}{\frac{3}{2}\pi}=\frac{2}{3}\times\frac{15}{8}\times\frac{2}{3}a$$

$$=\frac{5}{6}a.$$

4.9 Mass Centre of a Solid of Revolution

To find the centre of mass of a solid of revolution, i.e., a solid whose boundary is obtained by revolving a given plane curve, about a line in its plane.

Let the given curve be

$$y=f(x)$$

and the given line be taken as the x-axis (Fig. 4·24). Consider the variation of x from $x=a$ to $x=b$.

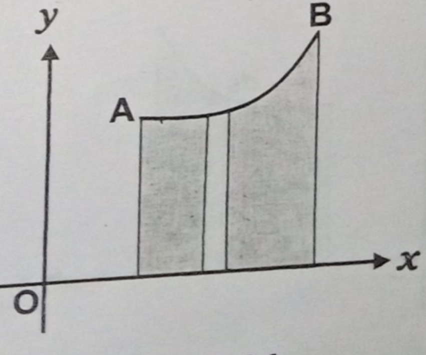

Fig. 4.24

The solid can be regarded as made up of thin circular slices perpendicular to the x-axis. The mass of the slice at a distance x from the origin is $\rho\pi y^2\delta x$. Since the c.m.

of each slice lies on the x-axis the *c.m.* of the whole solid also lies on the same line, *i.e.*,

$$\bar{y}=0.$$

Also $$\bar{x}=\frac{\int_a^b x.\rho\,\pi\,y^2\,dx}{\int_a^b \rho\,\pi\,y^2\,dx}=\frac{\int_a^b x\,y^2\,dx}{\int_a^b y^2\,dx}$$ (when ρ is constant).

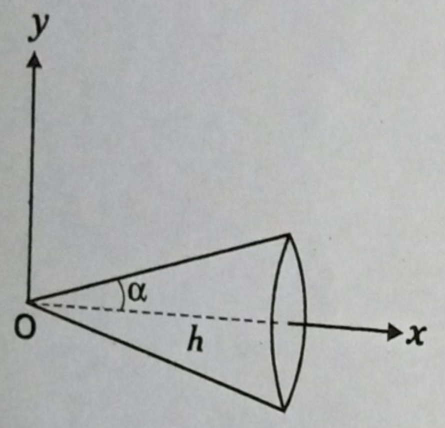

Fig. 4.25

Example. *A solid right circular cone.*

Sol. A solid right circular cone can be regarded as a solid of revolution formed by the rotation of the line

$$y=\tan\alpha.x, \qquad (0 \leqslant x \leqslant h)$$

about the x-axis (Fig. 4·25).

$$\therefore \quad \bar{x}=\frac{\int_0^h x\,y^2\,dx}{\int_0^h y^2\,dx}=\frac{\int_0^h x^3\tan^2\alpha\,dx}{\int_0^h x^2\tan^2\alpha\,dx}$$

$$=\frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx}=\frac{\left|\frac{x^4}{4}\right|_0^h}{\left|\frac{x^3}{3}\right|_0^h}$$

$$=\frac{3}{4}h.$$

4.10 Mass Centre of a Surface of Revolution

To find the c.m. of a surface of revolution, i.e., surface formed by the rotation of a plane curve about a line in its plane.

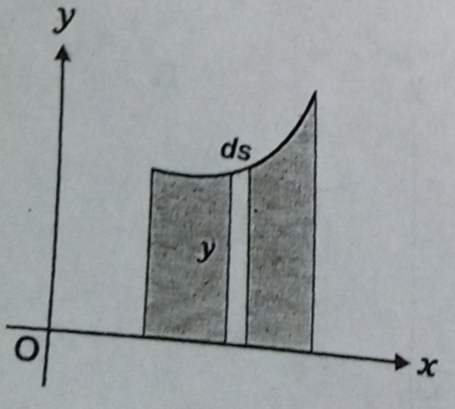

Fig. 4.26

Let the plane curve be $y=f(x)$ where x varies from $x=a$ to $x=b$, and let the rotation axis be taken as the x-axis (*see* Fig. 4·26). Then the area of an element of the surface of revolution is $2\pi y\,\delta s$.

Clearly $\bar{y}=0$.

Also $\bar{x}=\dfrac{\int \rho\, x\, 2\pi y\, ds}{\int \rho\, 2\pi y\, ds}$

$=\dfrac{\int \rho\, x\, y\, ds}{\int \rho\, y\, ds}$

$=\dfrac{\int x\, y\, ds}{\int y\, ds}$ (when the surface density is constant).

N.B. In a numerical case the value of ds in terms of Cartesian coordinates $\left(ds=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx\right)$, or polar coordinates $\left(ds=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\, d\theta\right)$, or a parameter $\left(ds=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\, dt\right)$ may be substituted as found convenient.

Example 1. Find the *c. m.* of a hollow right circular cone of semi-vertical angle α and height h.

Sol. A hollow right circular cone may be regarded as the surface of revolution generated by the rotation of the line

$$y=\tan \alpha . x \qquad (0 \leqslant x \leqslant h)$$

about the x-axis (Fig. 4·27).

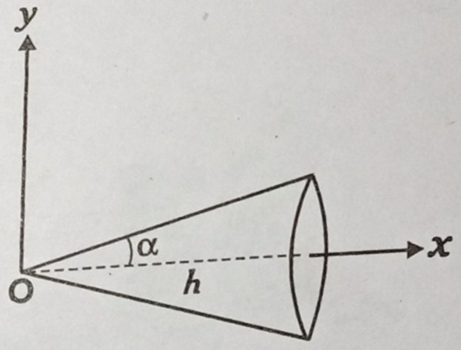

Fig. 4.27

By symmetry $\bar{y}=0$.

Also $\bar{x}=\dfrac{\int x\, y\, ds}{\int y\, ds}$

Now $\dfrac{dy}{dx}=\tan \alpha$.

$\therefore$ $ds=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx=\sqrt{1+\tan^2 \alpha}\, dx.$

$=\sec \alpha . dx$

$\therefore$ $\bar{x}=\dfrac{\int_0^h x . x \tan \alpha . \sec \alpha\, dx}{\int_0^h x \tan \alpha . \sec \alpha\, dx}$

$=\dfrac{\int_0^h x^2\, dx}{\int_0^h x\, dx}=\dfrac{2}{3} h.$

Example 2. Find the *c.m.* of the surface generated by the revolution of the arc of the parabola, lying between the vertex and the latus rectum, about the *x*-axis.

Sol., Let the parabola be

$$y^2 = 4a\,x,$$

so that $y = 2\sqrt{a}\sqrt{x}.$

$$\frac{dy}{dx} = 2\sqrt{a}\cdot\frac{1}{2\sqrt{x}} = \sqrt{\frac{a}{x}}.$$

$$\therefore \quad ds = \sqrt{1+\frac{a}{x}}\,dx.$$

By symmetry $\bar{y} = 0.$

$$\bar{x} = \frac{\int x\,y\,ds}{\int y\,ds} = \frac{\int_0^a x.\,2\sqrt{ax}.\,\sqrt{1+\frac{a}{x}}\,dx}{\int_0^a 2\sqrt{ax}\,\sqrt{1+\frac{a}{x}}\,dx}$$

$$= \frac{\int_0^a x\sqrt{a+x}\,dx}{\int_0^a \sqrt{a+x}\,dx}$$

$$= \frac{\int_0^a (x+a-a)\sqrt{x+a}\,dx}{\int_0^a \sqrt{x+a}\,dx}$$

$$= \frac{\int_0^a \left[(x+a)^{\frac{3}{2}} - a\sqrt{x+a}\right]dx}{\int_0^a \sqrt{x+a}\,dx}$$

$$= \frac{\left| \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2}{3}a(x+a)^{\frac{3}{2}} \right|_0^a}{\left| \frac{2}{3}(x+a)^{\frac{3}{2}} \right|_0^a}$$

$$= \frac{10 + 6\sqrt{2}}{35}$$

4.11 Theorems of Pappus

In certain cases in which we know the area of a surface or the volume of a solid, we can easily compute the position of the mass-centre with the help of two theorems called Theorems of Pappus.

Theorem 1. *Let there be a uniform distribution of mass along a plane curve C of length s lying entirely on one side of a line l which lies in the plane of the curve, and let p be the distance of the mass-centre of C from l. If S is the surface area generated by the rotation of C about l, then*

$$2\pi ps = S,$$

i.e., $$p = \frac{S}{2\pi s}.$$

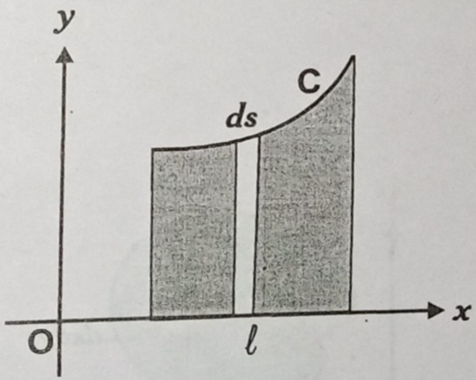

Fig. 4.28

Proof. We choose the coordinate system such that l is the x-axis (Fig. 4·28). In such a case,

$$p = \bar{y} \text{ for } C$$

$$= \frac{\int \rho y \, ds}{\int \rho \, ds} = \frac{\int y \, ds}{s}, \quad (\because \rho \text{ is constant})$$

$$\therefore \quad \int y ds = ps. \qquad \ldots (4{\cdot}20)$$

Also S = surface area generated by rotation of C about l (the x-axis)

$$= \int 2\pi y \, ds$$

$$= 2\pi \int y \, ds$$

$$= 2\pi ps, \qquad \text{(by 4·20)}$$

which proves the theorem.

Example. *A semi-circular wire.*

Sol. We know that the surface area of a sphere of radius a is

$$S=4\pi a^2.$$

But the sphere can be regarded as the surface of revolution generated by the rotation of a semi-circle about the diameter joining its ends. The length of the semi-circle is

$$s=\pi a.$$

$\therefore$ distance of the centroid (or *c.m.* in case of uniform semi-circular wire) of the semi-circle from the diameter is

$$p=\frac{S}{2\pi s}=\frac{4\pi a^2}{2\pi.\pi a}=\frac{2a}{\pi}.$$

Theorem 2. *Let there be a uniform distribution of mass on a plane region R, lying entirely on one side of a line l in the plane of the region. Let p be the distance of the mass-centre of R from l, and A the area of R. If V is the volume generated by the rotation of R about l, then*

$$2\pi p A=V,$$

i.e.,
$$p=\frac{V}{2\pi A}.$$

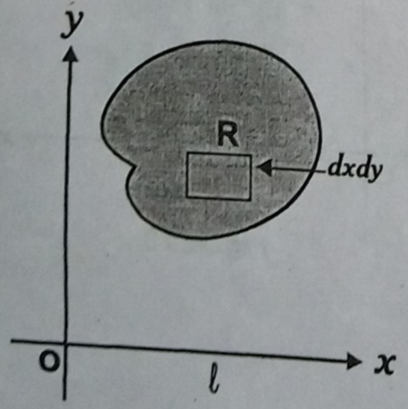

Fig. 4.29

Proof. We choose the coordinate system such that l is the x-axis and the plane of the region R is xy-plane (Fig. 4·29).

An area-element in R is $dx\,dy$

$\therefore$ $p=\bar{y}$ for R

$$=\frac{\int \rho\, y\, dx\, dy}{\int \rho\, dx\, dy}=\frac{\int y\, dx\, dy}{\int dx\, dy}, \quad (\because \rho \text{ is constant})$$

$$=\frac{\int y\, dx\, dy}{A}, \quad \left(\because \int dx\, dy=A\right)$$

$$\therefore \int y\, dx\, dy=A\,p. \quad \ldots (4{\cdot}21)$$

Now
$$V=\int 2\pi y\, dx\, dy$$
$$=2\pi \int y\, dx\, dy$$
$$=2\pi p A. \quad \text{(by 4·21)}$$

This proves the theorem.

Example. *A semi-circular plane lamina.*

Sol. We know that the volume of a solid sphere of radius a is

$$V = \frac{4}{3}\pi a^3.$$

But the sphere can be regarded as a solid of revolution generated by the rotation of a semi-circular lamina about its bounding diameter. The area of the semi-circular lamina is

$$A = \tfrac{1}{2}\pi a^2.$$

Therefore, distance of the centroid (*c.m.*) of the lamina from the diameter is

$$p = \frac{V}{2\pi A} = \frac{\frac{4}{3}\pi a^3}{2\pi \frac{1}{2}\pi a^2} = \frac{4a}{3\pi}.$$

We have seen (*see* Art. 2·15) that a system of parallel forces either has a resultant or reduces to a couple. In the former case the resultant passes through a particular point (the centre of parallel forces) for all the orientations of the forces provided their magnitudes and points of application are kept unchanged.

In the following article we discuss an important case of parallel forces.

4.12 Centre of Gravity

The result proved in Art. 2·15 has an important application. In case of a body within the gravitational field of the earth, each element of matter in it is attracted towards the centre of the earth. The force of attraction exerted on an element is called its weight. If the size of the body is small as compared with that of the earth (as is the case in almost all practical cases), the forces on all the elements will be parallel (*see* Fig. 4·30) and will have a resultant, equal to the weight of the body. The resultant weight will, for all orientations of the body, pass through a particular point G of the body. This point is called the *centre of gravity* of the body.

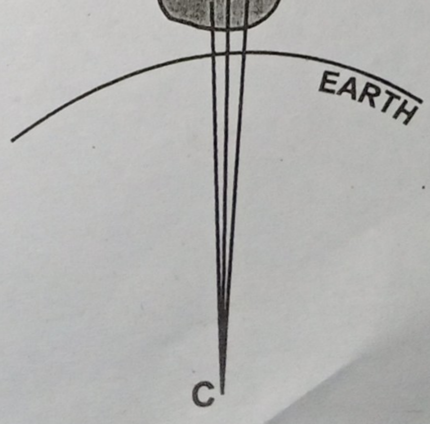

Fig. 4.30