Coordinate-free Form:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A}_{dip}(\vec{r})$$

$$= \frac{\mu_0}{4\pi}\,\vec{\nabla}\times\left(\vec{m}\times\frac{\vec{r}}{r^3}\right)$$

Using the vector identity,

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

where, $\vec{A}$ and $\vec{B}$ are any two vector functions.

(∵ $\vec{m}$ is a constant vector)

$$\vec{\nabla}\times\left(\vec{m}\times\frac{\vec{r}}{r^3}\right) = \cancelto{=0}{\left(\frac{\vec{r}}{r^3}\cdot\vec{\nabla}\right)\vec{m}} - (\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3} + \vec{m}\underbrace{\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)}_{=0.} - \frac{\vec{r}}{r^3}\cancelto{=0}{(\vec{\nabla}\cdot\vec{m})}$$

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot \frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

OR $$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{1}{r^3}\underbrace{(\vec{\nabla}\cdot\vec{r})}_{=3} + \vec{r}\cdot\left(\vec{\nabla}\frac{1}{r^3}\right)$$ $\Rightarrow$ vector identity

Now, $$\left(\vec{\nabla}\frac{1}{r^3}\right) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)(x^2+y^2+z^2)^{-3/2}$$

As, $$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} = -\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) = -\frac{3x}{r^5}$$

$$\therefore\ \vec{\nabla}\frac{1}{r^3} = -\frac{3}{r^5}(x\hat{x}+y\hat{y}+z\hat{z}) = -\frac{3\vec{r}}{r^5}$$

$$\therefore\ \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{3}{r^3} - \frac{3\underbrace{(\vec{r}\cdot\vec{r})}_{=r^2}}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

$$\therefore\ \vec{B}_{dip}(\vec{r}) = -\frac{\mu_0}{4\pi}(\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3}$$

Take in general, $\vec{m} = m_x\hat{x} + m_y\hat{y} + m_z\hat{z}$ (Although here, $\vec{m} = m_z\hat{z}$)

$$\vec{B}_{dip}(\vec{r}) = -\frac{\mu_0}{4\pi}\left(m_x\frac{\partial}{\partial x} + m_y\frac{\partial}{\partial y} + m_z\frac{\partial}{\partial z}\right)\frac{\vec{r}}{r^3}$$

Consider, $$m_x\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^3}\right) = m_x\frac{\partial}{\partial x}\left[\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= m_x\left[(x^2+y^2+z^2)^{-3/2}\hat{x} - \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}x)(x\hat{x}+y\hat{y}+z\hat{z})\right]$$

$$= m_x\left[\frac{1}{r^3}\hat{x} - \frac{3x}{r^5}\vec{r}\right]$$

Similarly,
$$m_y\frac{\partial}{\partial y}\left(\frac{\vec{r}}{r^3}\right) = m_y\left[\frac{1}{r^3}\hat{y} - \frac{3y}{r^5}\vec{r}\right]$$

and $$m_z\frac{\partial}{\partial z}\left(\frac{\vec{r}}{r^3}\right) = m_z\left[\frac{1}{r^3}\hat{z} - \frac{3z}{r^5}\vec{r}\right]$$

$$\therefore\ \vec{B}_{dip}(\vec{r}) = -\frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\underbrace{(m_x\hat{x} + m_y\hat{y} + m_z\hat{z})}_{=\vec{m}} - \frac{3}{r^5}\underbrace{(m_x x + m_y y + m_z z)}_{=\vec{m}\cdot\vec{r}}\vec{r}\right]$$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{r})\hat{r}}{r^3} - \frac{\vec{m}}{r^3}\right]$$

$$\Rightarrow \quad \underbrace{\boxed{\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]}}$$

coordinate free form.

compare with the electric field of an electric dipole.

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

As, $\vec{m} = m\hat{z} = m(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$\vec{r} = r\hat{r}$

$$\vec{m}\cdot\vec{r} = mr\cos\theta\,\underbrace{\hat{r}\cdot\hat{r}}_{=1} - mr\sin\theta\,\underbrace{\hat{\theta}\cdot\hat{r}}_{=0} = mr\cos\theta$$

or $\vec{m}\cdot\hat{r} = m\cos\theta\,\hat{r}\cdot\hat{r} - m\sin\theta\,\hat{\theta}\cdot\hat{r} = m\cos\theta$

$\therefore$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m\cos\theta\,\hat{r} - (m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta})\right]$$

$$\underbrace{\vec{B}_{dip}(\vec{r})} = \frac{\mu_0}{4\pi}\underbrace{\frac{m}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}$$

in spherical coordinates.