SYMMETRY AND CENTRE OF MASS

Symmetry with respect to a point

A uniform body is symmetric w.r.t origin if and only if for every point (x, y, z) of the body (−x, −y, −z) is also a point of the body. (origin bisects the line segment joining the two points (x, y, z) and (−x, −y, −z))
Symmetry w.r.t a point O is called Central Symmetry and point O is called centre of symmetry.

Examples

1. Uniform circular wire or a circular lamina is symmetric w.r.t its geometric centre.
2. Uniform solid sphere or spherical shell is symmetric w.r.t its geometric centre.
3. A uniform rod is symmetric w.r.t its mid point.

Symmetry w.r.t. a line

A uniform body is symmetric w.r.t the z-axis if and only if for every point (x, y, z) of the body, the point (−x, −y, z) is also a point of the body

(Z-axis is the right bisector of the line segment joining the points (x, y, z) and $(-x, -y, z)$).

<u>Examples</u>

1. A uniform hollow or solid right circular cone is symmetric with respect to its axis.

2. A uniform circular is also symmetric w.r.t its axis.

3. A uniform lamina is symmetric with respect to the x-axis if and only if corresponding to every point (x, y) of the lamina, there exists a point $(x, -y)$ of the lamina.

4. A uniform elliptic lamina or an elliptic wire is symmetric w.r.t both the principal axes.

* Symmetry about a line is called axial symmetry and the line of Symmetry is called the axis of Symmetry.

<u>SYMMETRY W.R.T A PLANE</u>

A uniform body is symmetric w.r.t the xy-plane if and only if corresponding to every point (x, y, z) of the body there

exists a point $(x, y, -z)$ in the body.
(In such a case the xy-plane bisects the join of (x,y,z) and $(x,y,\bar{z})$ perpendicularly)

1. A uniform solid or hollow ellipsoid is symmetric w.r.t each of its principal planes.

2. A uniform sphere is symmetric w.r.t each of its diametrical planes (planes passing through the centre)

Theorem (a) If a body possesses central symmetry, the centre of symmetry is the centre of mass.

(b) If a body possesses axial symmetry, the centre of mass lies on the symmetry axis.

(c) If the body is symmetric w.r.t a plane, the centre of mass lies on that plane.

COR.(i) If a body is symmetric w.r.t each of two planes, its centre of mass lies on their line of intersection.

COR.(ii) If a body is symmetric w.r.t

each of three planes, their point of intersection is the c.m of the body.

In view of above theorem:

(1) The c.m of a uniform solid sphere is its geometric centre.

(2) The c.m of a uniform solid hemisphere lies on the radial segment perpendicular to the plane face. (The solid is symmetric w.r.t each plane passing through the radial segment).

(3) The c.m of an elliptic wire or elliptic lamina is its geometric centre. (Each principal axis is an axis of symmetry. Therefore c.m lies at the intersection of the principal axes.)

(4) The c.m of an isosceles triangle lies on the perpendicular from the vertex on the base.

CHAPTER 4

CENTRES OF MASS AND GRAVITY

THE concepts of centre of mass and centre of gravity of a set of particles or a rigid body are very useful in Mechanics. In this chapter we shall state some basic definitions concerning them alongwith the usual methods of computing them.

4.1 Centre of Mass of a Set of Particles

Definition 1. Consider a set of n particles of masses $m_1, m_2, \ldots, m_n$, situated at the points $P_1, P_2, \ldots, P_n$ whose position vectors relative to an origin O are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. Then the *linear moment* of the set of particles with respect to O is the vector

$$\sum_{i=1}^{n} m_i \mathbf{r}_i.$$

Thus the linear moment, with respect to the origin, of two particles of mass 1, 2 lb. respectively placed at the points **i**−2**j**, 3**i** is

$$1\,(\mathbf{i}-2\mathbf{j})+2(3\mathbf{i})$$

$$=7\mathbf{i}-2\mathbf{j}.$$

Definition 2. The *centre of mass* (shortly written *c.m.*) of a set of particles is the point with respect to which the linear moment of the set is zero.

To justify this definition we should, in fact, prove

Theorem 1. Every set of particles has one and only one centre of mass.

Proof. Let the particles $m_1, m_2, \ldots, m_n$ be located at the points $P_1(\mathbf{r}_1), P_2(\mathbf{r}_2), \ldots, P_n(\mathbf{r}_n)$.

Suppose $C\,(\bar{\mathbf{r}})$ is a *c.m.*

The position vector of $P_i(\mathbf{r}_i)$ relative to C is $\mathbf{r}_i-\bar{\mathbf{r}}$.

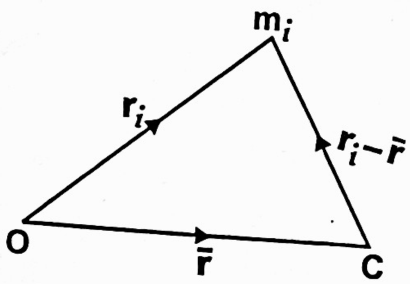

Fig. 4.1

$$\therefore \quad O=\Sigma\,{}^{*}m_i\,(\mathbf{r}_i-\bar{\mathbf{r}})$$

$$=\Sigma\, m_i\mathbf{r}_i-\Sigma\, m_i\,\bar{\mathbf{r}}$$

$$=\Sigma\, m_i\,\mathbf{r}_i-\bar{\mathbf{r}}\,\Sigma\, m_i$$

$$\therefore \quad \bar{\mathbf{r}}=\frac{\Sigma\, m_i\,\mathbf{r}_i}{\Sigma\, m_i}. \tag{4.1}$$

Thus a *c. m.* C exists and its position vector is $\dfrac{\Sigma\, m_i\,\mathbf{r}_i}{\Sigma\, m_i}$.

Suppose $C'(\bar{\mathbf{r}}')$ is another *c. m.* of the set of particles. Then reasoning as above, the linear moment of the given set of particles with respect to C' is zero if and only if

$$\bar{\mathbf{r}}'=\frac{\Sigma\, m_i\,\mathbf{r}_i}{\Sigma\, m_i}.$$

$$\therefore \quad \bar{\mathbf{r}}'=\bar{\mathbf{r}},\ i.e.,\ C'\equiv C.$$

This proves the uniqueness of the *c. m.*

* Henceforth i will be presumed to vary from 1 to n unless otherwise stated.

Note 1. Cartesian coordinates of the c. m.

Let $\mathbf{r}_i = (x_i, y_i, z_i)$

and $\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z})$.

Then
$$\bar{x} = \frac{\Sigma\, m_i x_i}{\Sigma\, m_i},$$

$$\bar{y} = \frac{\Sigma\, m_i y_i}{\Sigma\, m_i},$$

$$\bar{z} = \frac{\Sigma\, m_i z_i}{\Sigma\, m_i}.$$

In case of a coplanar set of particles, the last coordinate of the *c. m.* may be ignored by choosing *x*-, *y*-axes in the plane of the set and in case of a collinear set only one coordinate will be sufficient.

Note 2. Centroid. The point

$$\frac{\Sigma\, \mathbf{r}_i}{n} = \left(\frac{\Sigma\, x_i}{n}, \frac{\Sigma\, y_i}{n}, \frac{\Sigma\, z_i}{n}\right)$$

is called the *centroid* of the points $P_1, P_2, \ldots, P_n$.

In case $m_1 = m_2 = \ldots = m_n$, it is easy to see that the *c. m.* of the particles is the same as the centroid of the points specifying their positions.

Note 3. If a set of particles consists of two particles m_1, m_2 only, then

$$\bar{\mathbf{r}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.$$

But this equation also gives the position vector of the point dividing the directed line segment from $\mathbf{r}_1$ to $\mathbf{r}_2$ in the ratio $m_2 : m_1$. Therefore, we conclude that the *c. m.* of two particles m_1, m_2 divides the directed line segment joining them in the ratio $m_2 : m_1$.

In case $m_1 = m_2 = m$, say,

$$\bar{\mathbf{r}} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}.$$

which, as expected in view of Note 2 above, is the centroid of the points $\mathbf{r}_1, \mathbf{r}_2$.

Example. 1. Find the centroid of the points $\mathbf{i}$, $2\mathbf{i} - \mathbf{j}$ and $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. If particles of mass 2, 4, 3 grams are placed respectively at these points, what will be their *c. m.*?

Sol. The position vector of the centroid

$$=\frac{\mathbf{i}+2\mathbf{i}-\mathbf{j}+3\mathbf{i}+\mathbf{j}-4\mathbf{k}}{3}$$

$$=\frac{6\mathbf{i}-4\mathbf{k}}{3}.$$

The position vector of the *c. m.* is given by

$$\bar{\mathbf{r}}=\frac{2(\mathbf{i})+4(2\mathbf{i}-\mathbf{j})+3(3\mathbf{i}+\mathbf{j}-4\mathbf{k})}{2+4+3}$$

$$=\frac{19\mathbf{i}-\mathbf{j}-12\mathbf{k}}{9}.$$

The coordinates of the *c. m.* are $\left(\frac{19}{9}, -\frac{1}{9}, -\frac{12}{9}\right)$

4.2 Centre of Mass of a Continuous Distribution of Matter

The formulae obtained in the preceding article are applicable in the case of discrete set of particles only. If we are to find the *c. m.* of a continuous distribution of matter forming a body, integration methods explained below are to be employed.

Centre of mass of a thin rod

By a thin rod we mean a rigid body which has length only and whose breadth and thickness are negligible.

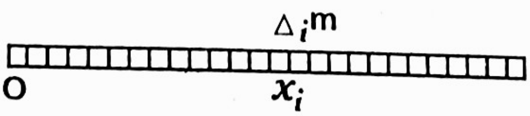

Fig. 4.2

Let m be the mass of a thin rod of length l. We subdivide the rod into n parts, and suppose that $\Delta_i m$ is the mass of the ith part. Let x_i be the distance from one end O of any point of the ith part (Fig. 4·2). Then the position of the *c. m.* is given by

$$\bar{x}=\underset{\substack{n\to\infty \\ max\,\Delta m\to 0}}{Lt}\frac{\Sigma\,\Delta_i m\, x_i}{\Sigma\,\Delta_i m}. \tag{4.2}$$

Clearly the expression on the left may be written in the form of integral so that

$$\bar{x}=\frac{\int_0^l x\,dm}{\int_0^l dm}=\frac{\int_0^l x\,dm}{m}. \tag{4.3}$$

If the density of the rod at the point x is ρ,

$$\bar{x}=\frac{\int_0^l x\rho\, dx}{\int_0^l \rho\, dx}=\frac{\int_0^l \rho x\, dx}{m}. \quad (4.4)$$

In case of a rod of uniform density (*homogeneous rod*)

$$\bar{x}=\frac{\int_0^l x\, dx}{\int_0^l dx} \quad (4.5)$$

$$=\frac{\frac{l^2}{2}}{l}$$

$$=\frac{l}{2}. \quad (4.6)$$

N.B. Equation (4·5) defines the *centroid* of the rod. Therefore, in the case of a homogeneous rod, its *c. m.* is the same as its centroid.

Centre of mass of a lamina

By a lamina we mean a flat rigid body whose thickness is negligible. Let m be the mass of a given lamina and A its area. We subdivide the lamina into rectangles* by drawing lines parallel to the coordinate axes (Fig. 4·3). We number the complete rectangles lying within the area A from 1 to n.

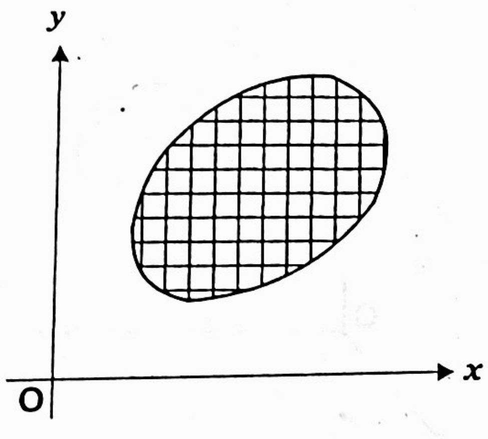

Fig. 4.3

Let $\Delta_i m$ be the mass of the ith rectangle whose diagonal is of length $\delta_i l$ and let $\mathbf{r}_i=(x_i, y_i)$ be any point in it. Then the *c. m.* of the lamina is given by

$$\bar{\mathbf{r}}=\lim_{\substack{n\to\infty \\ max\, \delta_i l\to 0}} \frac{\sum_{i=1}^{n} \mathbf{r}_i \Delta_i m}{\sum_{i=1}^{n} \Delta_i m}. \quad (4.7)$$

The left hand member of this equation can be written in the form of integral so that

* By 'a rectangle' we mean a rectangular region.

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\,dm}{\int dm}=\frac{\int \mathbf{r}\,dm}{m}, \tag{4.8}$$

where the integration has to be performed over the entire lamina.

If the density of the lamina at $\mathbf{r}$ is ρ, then $dm=\rho dA$ and the last equation takes the form

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\rho\,dA}{\int \rho\,dA}=\frac{\int \rho\mathbf{r}\,dA}{m} \tag{4.9}$$

In case of a homogeneous lamina

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\,dA}{\int dA}=\frac{\int \mathbf{r}\,dA}{A}. \tag{4.10}$$

The last equation also defines the centroid of the lamina. Therefore, in the case of a homogeneous lamina the *c. m.* is the same as its centroid.

In terms of Cartesian coordinates,

$$\mathbf{r}=(x, y) \text{ and } dA=dx\,dy.$$

Therefore, equation (4·9) takes the form

$$m\,\bar{x}=\iint \rho x\,dx\,dy,\; m\,\bar{y}=\int \rho\,y\,dx\,dy. \tag{4.9}$$

Centre of mass of a solid

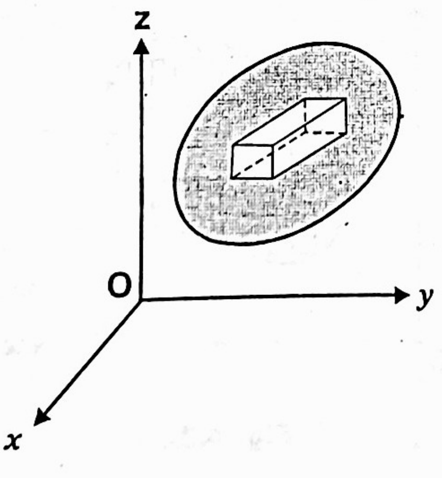

Fig. 4.4

Suppose we are to find the *c.m.* of a three-dimensional rigid body whose mass is m and volume is V. We subdivide the body into rectangular parallelepipeds* by drawing planes parallel to the coordinate planes. One such parallelepiped is shown in Fig. 4·4. We number the complete parallelepipeds within V from 1 to n.

Let $\Delta_i m$ be the mass of the ith parallelepiped, whose diagonal is of length $\delta_i l$ and $\mathbf{r}_i=(x_i, y_i, z_i)$ be any point within it. Then

$$\bar{\mathbf{r}}=\underset{\substack{n\to\infty \\ max\,\delta_i l\to 0}}{Lt}\frac{\sum_{i=1}^{n}\mathbf{r}_i\Delta_i m}{\sum_{i=1}^{n}\Delta_i m}. \tag{4.11}$$

---

*By a 'rectangular parallelepiped' we mean here a region of space bounded by a rectangular parallelepiped.

The *R.H.S.* can be written in the form of integrals so that

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\,dm}{\int dm}=\frac{\int \mathbf{r}\,dm}{m}, \tag{4.12}$$

where the integration has to be performed over the entire solid.

If the density of the solid at $\mathbf{r}$ is ρ, then

$$dm=\rho\,dV,$$

where dV is the volume element, and the last equation assumes the form

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\rho\,dV}{\int \rho\,dV}=\frac{\int \rho\mathbf{r}\,dV}{m}. \tag{4.13}$$

In the case of a uniform solid.

$$\bar{\mathbf{r}}=\frac{\int \mathbf{r}\,dV}{\int dV}=\frac{\int \mathbf{r}\,dV}{V}. \tag{4.14}$$

The last equation also defines the centroid of the solid. Therefore, in the case of a homogeneous solid the *c. m.* coincides with the centroid.

In terms of Cartesian coordinates

$$\mathbf{r}=(x, y, z),\ dV=dx\,dy\,dz.$$

∴ equation (4·13) gives

$$\left.\begin{aligned} m\bar{x}&=\iiint \rho x\,dx\,dy\,dz,\\ m\bar{y}&=\iiint \rho y\,dx\,dy\,dz,\\ m\bar{z}&=\iiint \rho z\,dx\,dy\,dz.\end{aligned}\right\}. \tag{4.13}$$

N.B. Comparing formula (4·1) with formulae (4·3), (4·8) and (4·12) we notice that the latter are similar to the former except that in place of the summation sign we have the integration sign and in place of m_i we have dm (the mass element) in the latter formulae.

4.3 Symmetry and Centre of Mass

If a body possesses some sort of symmetry, much of the labour involved in computing the position of its centre of mass can be avoided. We first explain the concept of symmetry and shall thereafter show how to use this concept in determining the *c. m.* of a body.

Symmetry with respect to a point

A body is said to be symmetric with respect to a point O if and only if corresponding to every point P of the body there exists a point P' in the body such that O is the middle point of the line segment PP' and $\rho(P)=\rho(P')$, *i.e.*, the density of the body at the points P, P' is the same.

From this definition it follows that a *uniform* body is symmetric with respect to the origin if and only if for every point(x,y,z) of the body $(-x, -y, -z)$ is also a point of the body. (Notice that the origin bisects the line segment joining the points (x,y,z) and $(-x, -y, -z)$).

Symmetry with respect to a point O is called *central symmetry* and the point O is called the *centre of symmetry.*

A uniform circular wire or a circular lamina is symmetric with respect to its geometric centre.

Similarly a uniform solid sphere or spherical shell is symmetric with respect to its geometric centre.

A uniform rod is symmetric with respect to its mid-point.

Symmetry with respect to a line

A body is said to be symmetric with respect to a line l if and only if corresponding to every point P of the body there exists a point P' in the body such that l bisects PP' perpendicularly and $\rho(P)=\rho(P')$.

In particular, a *uniform* body is symmetric with respect to the z-axis if and only if for every point (x,y,z) of the body, the point$(-x, -y, z)$ is also a point of the body. (Notice that the z-axis is the right bisector of the line segment joining the points (x,y,z) and $(-x, -y, z)$.

A uniform hollow or solid right circular cone is symmetric with respect to its axis.

A uniform circular cylinder is also symmetric with respect to its axis.

A uniform lamina is symmetric with respect to the x-axis if and only if corresponding to every point (x, y) of the lamina, there exists a point $(x,-y)$ of the lamina. Similarly or symmetry with respect to the y-axis.

A uniform elliptic lamina or an elliptic wire is symmetric with respect to both the principal axes.

Symmetry about a line is called *axial symmetry* and the line of symmetry is called the *axis of symmetry*.

Symmetry with respect to a plane

A body is said to be symmetric with respect to a plane p if and only if corresponding to every point P of the body there exists a point P' in the body such that p bisects PP' perpendicularly and $\rho(P)=\rho(P')$.

In particular, a uniform body is symmetric with respect to the xy-plane if and only if corresponding to every point (x, y, z) of the body there exists a point $(x, y, -z)$ in the body. (Notice that in such a case the xy-plane bisects the join of (x, y, z) and $(x, y, -z)$ perpendicularly).

A uniform solid or hollow ellipsoid is symmetric with respect to each of its principal planes.

A uniform sphere is symmetric with respect to each of its diametral planes (*i.e.*, planes passing through the centre).

We are now in a position to prove

Theorem. (*a*) If a body possesses central symmetry, the centre of symmetry is the centre of mass.

(*b*) If a body possesses axial symmetry, the centre of mass lies on the symmetry axis.

(*c*) If a body is symmetric with respect to a plane, the centre of mass lies on that plane.

Proof. (*a*) We take the origin of coordinates at the centre of symmetry. Then

$$M\bar{x}=\iiint \rho(x, y, z)\, x\,dV.$$

The integral on the right hand side consists of pairs of elements of the type $\rho(x, y, z)x\,dV$ and $-\rho(-x, -y, -z)x\,dV$ which are equal in magnitude but opposite in sign.

$\therefore$ the integral vanishes and so $\bar{x}=0$.

Similarly $\bar{y}=0$, $\bar{z}=0$.

Parts (*b*) and (*c*) may be proved in a similar manner.

Cor. (*i*) : If a body is symmetric with respect to each of two planes, its centre of mass lies on their line of intersection.

Cor. (*ii*) : If a body is symmetric with respect to each of three planes, their point of intersection is the centre of mass of the body.

In view of the above theorem :

(1) The *c.m.* of a uniform solid sphere is its geometric centre.

(2) The *c.m.* of a uniform solid hemisphere lies on the radial segment perpendicular to the plane face. (The solid is symmetric with respect to each plane passing through the radial segment).

(3) The *c.m.* of an elliptic wire or elliptic lamina is its geometric centre. (Each principal axis is an axis of symmetry. Therefore, the *c.m.* lies at the intersection of the principal axes.)

(4) The *c.m.* of an isosceles triangle lies on the perpendicular from the vertex on the base.

4.4 The Centre of Mass of the Union of any Number of Disjoint Sets of Particles

Suppose a set S of particles is the union of two disjoint sets of particles

$$S_1 : m_1, m_2, \ldots, m_{n_1},$$

$$S_2 : m_{n_1+1}, m_{n_1+2}, \ldots, m_n.$$

Let $\bar{\mathbf{r}}$ be the *c.m.* of $S=S_1 \cup S_2$, $\bar{\mathbf{r}}_1$ that of S_1 and $\bar{\mathbf{r}}_2$ that of S_2.

Then
$$\bar{\mathbf{r}}=\frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i}=\frac{\sum_{i=1}^{n_1} m_i \mathbf{r}_i+\sum_{i=n_1+1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n_1} m_i+\sum_{i=n_1+1}^{n} m_i}$$

$$=\frac{M_1 \bar{\mathbf{r}}_1+M_2 \bar{\mathbf{r}}_2}{M_1+M_2}, \qquad \ldots (4{\cdot}15)$$

where $M_1 = \sum_{i=1}^{n_1} m_i, \quad M_2 = \sum_{i=n_1+1}^{n} m_i$

are the total masses of the particles of S_1 and S_2 respectively.

Equation (4·15) shows that the *c.m.* of S divides the join of the centres of mass of S_1 and S_2 in the ratio $M_2 : M_1$.

This result can easily be extended to the case where S is the union of any number of disjoint sets

$$S_1, S_2, S_3, \ldots.$$

with total masses

$$M_1, M_2, M_3, \ldots.$$

In such a case

$$\bar{\mathbf{r}} = \frac{M_1 \bar{\mathbf{r}}_1 + M_2 \bar{\mathbf{r}}_2 + M_3 \bar{\mathbf{r}}_3 + \ldots}{M_1 + M_2 + M_3 + \ldots}. \qquad \ldots (4·16)$$

This result may be stated in words as follows :

If a set of particles S is the union of a number of disjoint sets of particles $S_1, S_2, \ldots$ with total masses $M_1, M_2, \ldots,$ then the *c.m.* of S is the same as that of a set of particles $M_1,$ $M_2, \ldots$ placed respectively at the centres of mass of $S_1, S_2, \ldots$

4.5 To Find the c.m of the Set of Particles S—S₁ When the Centres of Mass of S and S₁ are Known.

Suppose we know the position of the *c.m.* of a set of particles S, and also the position of the *c.m.* of a set of particles S_1 where $S_1 \subset S$, and we are to find the position of the *c.m.* of the set of particles $S - S_1$.

Let $S - S_1 = S_2$.

From result (4·15) of the preceding article,

$$M \bar{\mathbf{r}} = M_1 \bar{\mathbf{r}}_1 + M_2 \bar{\mathbf{r}}_2,$$

i.e., $M_2 \bar{\mathbf{r}}_2 = M \bar{\mathbf{r}} - M_1 \bar{\mathbf{r}}_1.$

$$\therefore \quad \bar{\mathbf{r}}_2 = \frac{M\bar{\mathbf{r}} - M_1 \bar{\mathbf{r}}_1}{M_2}$$

$$= \frac{M\bar{\mathbf{r}} - M_1 \bar{\mathbf{r}}_1}{M - M_1}, \qquad \ldots (4.17)$$

$$(\because \quad M_1 + M_2 = M)$$

which gives the required *c.m.*

This result shows that the *c.m.* of $S-S_1$ divides the join of the centres of mass of S and S_1 externally in the ratio $M_1 : M$.

The result can be easily extended.

If $S_1, S_2, \ldots,$ are disjoint proper sub-sets of a set of particles S with total masses $M_1, M_2, \ldots,$ then the *c.m.* of the set of particles

$$S - S_1 - S_2 \ldots$$

has the position vector

$$\frac{M\bar{\mathbf{r}} - M_1 \bar{\mathbf{r}}_1 - M_2 \bar{\mathbf{r}}_2 - \ldots}{M - M_1 - M_2 - \ldots}. \qquad \ldots (4.18)$$

N.B. It is easy to see that the results of this article and the preceding article are valid when in place of sets of particles $S, S_1, S_2, \ldots,$ we have non-overlapping rigid bodies with masses $M, M_1, M_2, \ldots$

By $S_1 + S_2$ we shall mean a compound body having two parts S_1 and S_2. Similarly $S-S_1$ will stand for the remainder of a body S from which a part S_1 has been removed.

Example. In a uniform circular disc of 8″ radius a circular hole of 2″ radius is cut, the centre of the hole being 3″ from the centre of the disc. Find the centre of mass of the remainder of the disc.

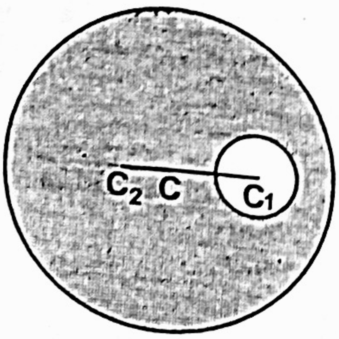

Fig. 4.5

Sol. Let M, M_1 be the masses and C, C_1 the centres of the disc and the hole respectively (Fig. 4·5). Then the *c.m.*, C_2, of the remainder divides C_1C externally in the ratio $M : M_1$.

Taking C as the origin,

$$CC_2 = \frac{M.0 - M_1.3}{M - M_1} = \frac{-3M_1}{M - M_1}.$$

If ρ is the density of the disc,

$$M=\rho.\pi.64=64\pi\rho,$$

$$M_1=\rho.\pi.4=4\pi\rho.$$

$$\therefore \qquad CC_2=\frac{-12\pi\rho}{64\pi\rho-4\pi\rho}=-\frac{1}{5},$$

i.e., C_2 is $\frac{1}{5}''$ from C on the opposite side of C_1.

4.6 Examples

Use of the above principles is further illustrated by means of worked examples.

Example 1. *A thin uniform rod.*

Sol. Since a thin uniform rod is symmetric with respect to its middle point, the same point is its centre of mass (centroid).

The position of the *c.m.* was determined by integration method in Art: 4·3 above.

Example 2. *A uniform rectangular lamina.*

Sol. Since a rectangular lamina is symmetric with respect to the lines joining the mid-points of the pairs of opposite sides, the point of intersection of such lines is the *c.m.* of the lamina.

The result is clear from the fact that a rectangular lamina possesses central symmetry with regard to the point of intersection of its diagonals.

Example 3. *A uniform wire bent into the form of an isosceles right-triangle.*

Sol. Let ABC be the isosceles triangle with $\angle A$ a right angle. Then the masses of the portions AB, AC, BC are proportional to their lengths. Let these be $m, m, \sqrt{2}m$ respectively. The three portions can be replaced by particles of these masses placed at their mid-points $D\left(\frac{a}{2},0\right)$, $E\left(0,\frac{a}{2}\right)$, $F\left(\frac{a}{2},\frac{a}{2}\right)$, where a is the length of either of the equal sides (by Art: 4·4) and axes have been chosen as shown in Fig. 4·6.

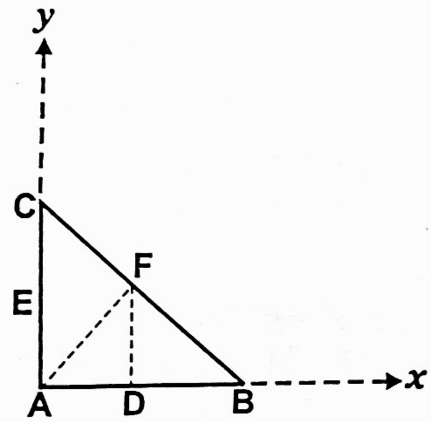

Fig. 4.6

Hence the coordinates of the *c.m.* of the whole wire are

$$\bar{x}=\frac{m.\frac{a}{2}+m.0+\sqrt{2}\,m.\frac{a}{2}}{m+m+\sqrt{2}\,m}=\frac{a(\sqrt{2}+1)}{2(2+\sqrt{2})}$$

$$=\frac{a}{2\sqrt{2}},$$

$$\bar{y}=\frac{m.0+m.\frac{a}{2}+m\sqrt{2}.\frac{a}{2}}{m+m+\sqrt{2}\,m}=\frac{a}{2\sqrt{2}}.$$

Example 4. *A uniform lamina whose boundary is a parallelogram.*

Sol. Let the boundary of the given lamina be the parallelogram *ABCD* as in Fig. 4·7. The lamina can be regarded as consisting of thin strips, such as *PQ*, parallel to *AB*. The *c.m.* of *PQ* lies on *EF*, the line segment joining mid-points of *AB* and *CD*. Each strip may, therefore, be replaced by a single particle having the same mass as the strip and placed at its mid-point. Since the *c.m.* of each strip lies on *EF*, the *c.m.* of the lamina must also be on *EF*. Similarly, if the lamina is considered to be made up of strips parallel to *AD*, its *c.m.* will be on *GH*, the line segment joining the mid-points of *AD* and *BC*.

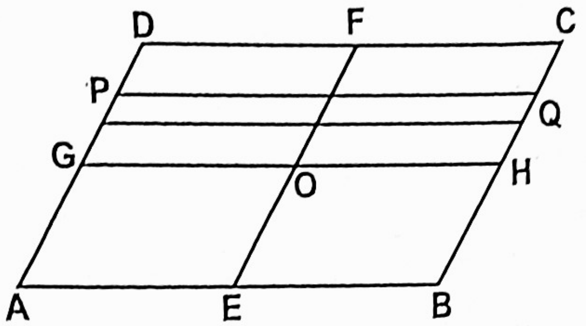

Fig. 4.7

Hence the *c.m.* of the lamina is *O*, the point of intersection of *EF* and *GH*.

The result is also evident from the fact that the lamina possesses central symmetry about *O*.

It is easy to see (geometrically) that *O* is also the point of intersection of the diagonals of the parallelogram.

Example 5. Find the mass-centre of a cubical box with no lid, the sides and bottom being made of the same thin material.

Sol. Let M be the mass of a box with lid and M_1 that of the lid.

By symmetry the *c.m.* of the given box lies on the vertical line through the centre. If the centre of the box is taken as the origin and the vertical through it as the *x*-axis, the *x*-coordinate of the *c.m.* is

$$\bar{x}=\frac{M.0-M_1.\frac{a}{2}}{M-M_1},$$

where a is the length of an edge of the box and $\frac{a}{2}$ is, therefore, the height of the open face from the centre of the cube.

$$\therefore \qquad \bar{x}=-\frac{a}{2}\frac{M_1}{M-M_1}.$$

Now $M=6\,M_1$.

$$\therefore \quad \bar{x}=-\frac{a}{2}\,\frac{M_1}{6M_1-M_1}=-\frac{a}{10}.$$

Thus the *c.m.* is at a distance $\frac{a}{10}$ below the centre. The height of the *c.m.* above the lower face is, therefore,

$$\frac{a}{2}-\frac{a}{10}=\frac{2}{5}\,a.$$

Example 6. *A uniform triangular lamina.*

Sol. Let ABC be the triangular lamina. It can be regarded as made up of thin strips such as $B'C'$, parallel to BC as in Fig. 4·8. The *c.m.* of $B'C'$ is at its mid-point which lies on the median AD. The *c.m.* of all other strips will also lie on the same median. Therefore, the *c.m.* of the entire lamina lies on AD. Supposing the lamina to be consisting of strips parallel to CA and AB, the *c.m.* would lie on each of the remaining medians BE and CF.

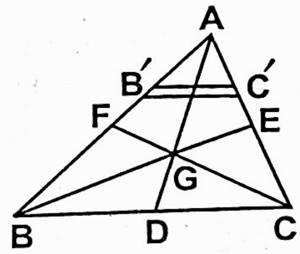

Fig. 4.8

Thus the *c.m.* of the triangular lamina, G, is the point of intersection of its medians.

Example 7. The *c.m.* of any uniform triangular lamina is the same as that of three equal particles placed at the vertices of the lamina.

Sol. This result follows easily from that proved in the preceding example.

Example 8. *Three uniform rods forming a triangular frame.*

Sol. Let BC, CA, AB be the rods forming the frame ABC (Fig. 4·9). Let D, E, F be their mid-points. If m is the mass per unit length of each rod, and a, b, c are their lengths, their masses are ma, mb, mc. The rods can be replaced by particles of the same masses, placed at the midpoints D, E, F. Suppose that with respect to a coordinate system in the plane of the frame, the coordinates of the midpoints of the rods are

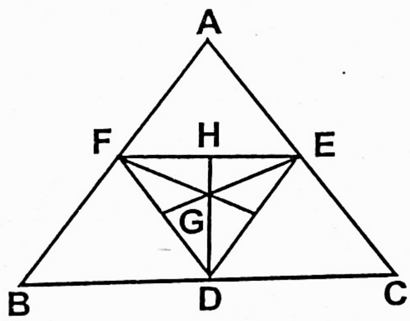

Fig. 4.9

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$$

$$\bar{y}=\frac{m.0+m.\frac{a}{2}+m\sqrt{2}.\frac{a}{2}}{m+m+\sqrt{2}\,m}=\frac{a}{2\sqrt{2}}.$$

Example 4. *A uniform lamina whose boundary is a parallelogram.*

Sol. Let the boundary of the given lamina be the parallelogram $ABCD$ as in Fig. 4·7. The lamina can be regarded as consisting of thin strips, such as PQ, parallel to AB. The *c.m.* of PQ lies on EF, the line segment joining mid-points of AB and CD. Each strip may, therefore, be replaced by a single particle having the same mass as the strip and placed at its mid-point. Since the *c.m.* of each strip lies on EF, the *c.m.* of the lamina must also be on EF. Similarly, if the lamina is considered to be made up of strips parallel to AD, its *c.m.* will be on GH, the line segment joining the mid-points of AD and BC.

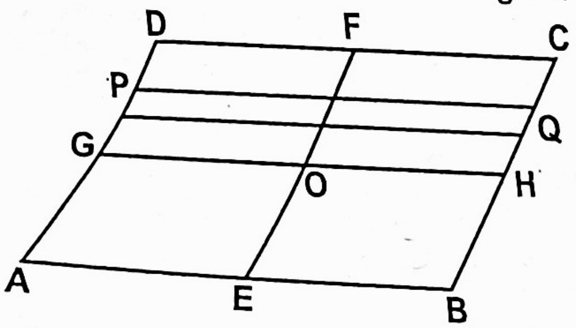

Fig. 4.7

Hence the *c.m.* of the lamina is O, the point of intersection of EF and GH.

The result is also evident from the fact that the lamina possesses central symmetry about O.

It is easy to see (geometrically) that O is also the point of intersection of the diagonals of the parallelogram.

Example 5. Find the mass-centre of a cubical box with no lid, the sides and bottom being made of the same thin material.

Sol. Let M be the mass of a box with lid and M_1 that of the lid.

By symmetry the *c.m.* of the given box lies on the vertical line through the centre. If the centre of the box is taken as the origin and the vertical through it as the x-axis, the x-coordinate of the *c.m.* is

$$\bar{x}=\frac{M.0-M_1.\frac{a}{2}}{M-M_1},$$

where a is the length of an edge of the box and $\frac{a}{2}$ is, therefore the height of the open face from the centre of the cube.

$$\therefore \quad \bar{x}=-\frac{a}{2}\frac{M_1}{M-M_1}.$$

Now $M=6\,M_1$.

$$\therefore \quad \bar{x}=-\frac{a}{2}\,\frac{M_1}{6M_1-M_1}=-\frac{a}{10}.$$

Thus the *c.m.* is at a distance $\frac{a}{10}$ below the centre. The height of the *c.m.* above the lower face is, therefore,

$$\frac{a}{2}-\frac{a}{10}=\frac{2}{5}\,a.$$

Example 6. *A uniform triangular lamina.*

Sol. Let ABC be the triangular lamina. It can be regarded as made up of thin strips such as $B'C'$, parallel to BC as in Fig. 4·8. The *c.m.* of $B'C'$ is at its mid-point which lies on the median AD. The *c.m.* of all other strips will also lie on the same median. Therefore, the *c.m.* of the entire lamina lies on AD. Supposing the lamina to be consisting of strips parallel to CA and AB, the *c.m.* would lie on each of the remaining medians BE and CF.

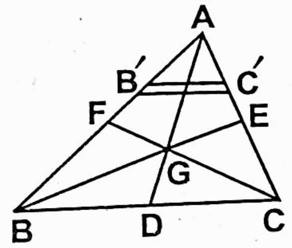

Fig. 4.8

Thus the *c.m.* of the triangular lamina, G, is the point of intersection of its medians.

Example 7. The *c.m.* of any uniform triangular lamina is the same as that of three equal particles placed at the vertices of the lamina.

Sol. This result follows easily from that proved in the preceding example.

Example 8. *Three uniform rods forming a triangular frame.*

Sol. Let BC, CA, AB be the rods forming the frame ABC (Fig. 4·9). Let D, E, F be their mid-points. If m is the mass per unit length of each rod, and a, b, c are their lengths, their masses are ma, mb, mc. The rods can be replaced by particles of the same masses, placed at the midpoints D, E, F. Suppose that with respect to a coordinate system in the plane of the frame, the coordinates of the midpoints of the rods are

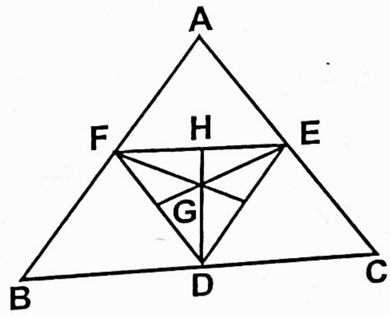

Fig. 4.9

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$$

respectively. Then the coordinates of the *c.m.* are

$$\bar{x}=\frac{ma\,x_1+mb\,x_2+mc\,x_3}{ma+mb+mc}=\frac{ax_1+bx_2+cx_3}{a+b+c},$$

$$\bar{y}=\frac{ma\,y_1+mb\,y_2+mc\,y_3}{ma+mb+mc}=\frac{ay_1+by_2+cy_3}{a+b+c}.$$

Thus the *c.m.* of the frame is the in-centre (the point of intersection of the angle-bisectors) of the $\triangle DEF$.

The result of this example may be stated as follows:

The *c.m.* of a uniform triangular frame is the same as that of three particles whose masses are proportional to the lengths of the sides and which are placed at their mid-points.

Example 9. Prove that the *c.m.* of a lamina bounded by a quadrilateral coincides with that of four equal particles placed at its vertices together with a fifth particle of equal but negative mass placed at the intersection of its diagonals.

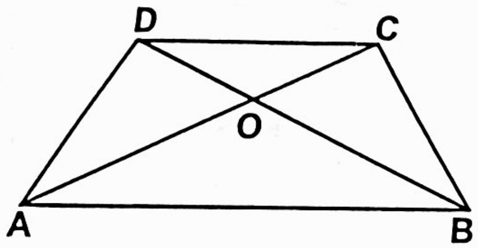

Fig. 4.10

Sol. Let a lamina be bounded by the quadrilateral *ABCD*, whose diagonals intersect at the point *O* (Fig. 4·10). Let *m, m'* be the masses of the triangular portions *ABD* and *CBD*. Since these triangles have the common base *BD*,

$$m : m' = AO : OC. \qquad \ldots(1)$$

Now the triangular lamina *ABD* has the same *c.m.* as three equal particles, each of mass $\frac{m}{3}$, placed at the vertices *A, B* and *D* respectively. Similarly the *c.m.* of the portion *BCD* coincides with that of three equal particles, each of mass $\frac{m'}{3}$, placed at *B, C* and *D* respectively. Hence the *c.m.* of the total lamina coincides with that of four particles

(*i*) $\frac{m}{3}$ at *A*,

(*ii*) $\frac{m+m'}{3}$ at *B*,

(*iii*) $\frac{m+m'}{3}$ at *D*,

(*iv*) $\frac{m'}{3}$ at *C*.

To make the four particles equal we may add particles of mass $\frac{m'}{3}$ at A and $\frac{m}{3}$ at C and counter-balance them by placing a particle of mass $-\frac{m+m'}{3}$ at the *c.m.* of these new particles.

Now the *c.m.* of particles of mass $\frac{m'}{3}$ at A and $\frac{m}{3}$ at C divides AC in the ratio $\frac{m}{3}:\frac{m'}{3}=m:m'$, *i.e.*, it coincides with the point O (by equation (1) above).

This proves the desired result.

Example 10. *A uniform tetrahedronal solid.*

Sol. We shall employ the following two geometrical properties of the tetrahedronal solid $ABCD$:

(1) Let $B'C'D'$ be the section of the solid parallel to the face BCD, and let G_1 be the centroid of the face BCD (Fig. 4·11). If AG meets the section $B'C'D'$ in G_1' then G_1' is the centroid of the triangular lamina $B'C'D'$.

(2) The line-segments (such as AG_1) joining the vertices of the solid to the centroids of the opposite triangular faces are concurrent and divide one another in the ratio 3 : 1.

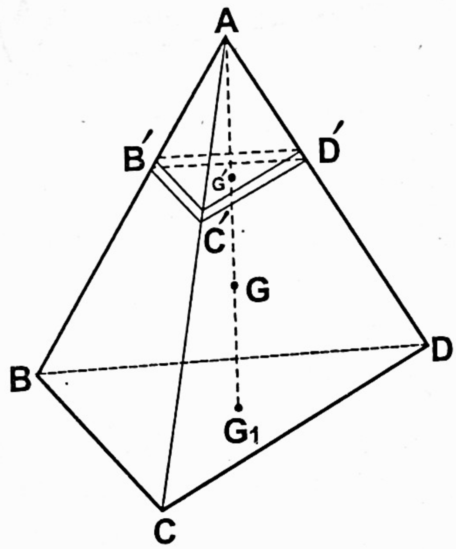

Fig. 4.11

The solid can be regarded as consisting of thin slices such as $B'C'D'$, parallel to the face BCD. The *c.m.* G_1', of the slice $B'C'D'$ lies on AG_1, the line segment joining A to the centroid of the face BCD. Therefore, the *c.m.* of the whole solid lies on AG_1. Similarly the *c.m.* lies on each of the other line segments joining the vertices of the solid to the centroids of the opposite faces. But all these line-segments are concurrent and divide one another in the ratio 3 : 1. Therefore, their point of concurrence is the desired *c.m.*

N.B. (1) If the length of the perpendicular from A to the face BCD be h (called the height of the solid), the height of G above the face BCD is $\frac{h}{4}$.

(2) The result of the example can be used to show that the *c.m.* of a pyramid having a polygonal base lies on the line joining its vertex to the centroid of the base and is at a height $\frac{h}{4}$ above the base, where h is the total height of the pyramid (Fig. 4·12).

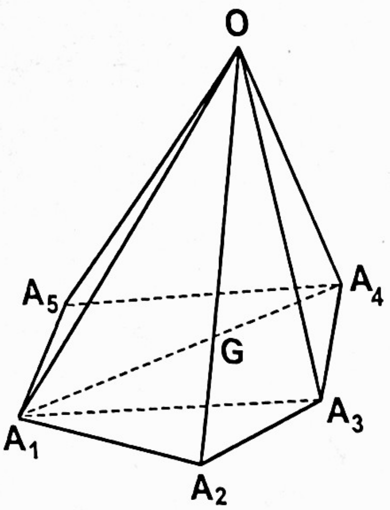

Fig. 4.12

Since a solid cone is a special case of a pyramid in which the base may be regarded as a polygon having infinitely many sides, its *c. m.* also lies at a height $\frac{h}{4}$ above the base.

Example 11. *A right circular solid cone.*

Sol. A right circular solid cone is symmetric with regard to the line joining its vertex to the centre of the base. This line is called the *axis* of the cone.

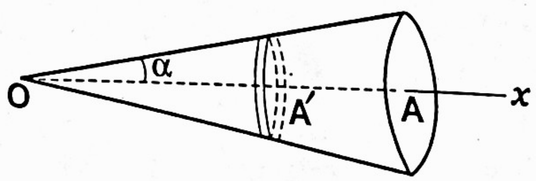

Fig. 4.13

We take O, the vertex of the given cone, as the origin and x-axis along the axis of the cone as in (Fig. 4·13).

The cone can be regarded as made up of thin circular slices parallel to the base. Consider a circular slice at distance x from O. If α is the measure of the semi-vertical angle, the radius of the slice is $x \tan \alpha$.

Therefore, area of the slice $= \pi x^2 \tan^2\alpha$.

Its volume $= \pi x^2 \tan^2\alpha . \delta x$,

where δx is its thickness.

Mass of the disc $= \rho \pi x^2 \tan^2\alpha . \delta x$,

where ρ is the density of the solid.

By symmetry the *c.m.* of the solid lies on its axis. The distance of the *c.m.* from the vertex is given by

$$\bar{x} = \frac{\int_0^h x \pi \rho x^2 \tan^2\alpha \, dx}{\int_0^h \pi \rho x^2 \tan^2\alpha \, dx},$$

where h is the total height of the cone.

$$= \frac{\int_0^h x^3 \, dx}{\int_0^h x^2 \, dx} = \left| \frac{x^4}{4} \cdot \frac{3}{x^3} \right|_0^h$$

$$= \tfrac{3}{4} h.$$

Thus the *c.m.* of a right circular cone of height h is at a distance $\frac{3}{4}h$ from the vertex along the symmetry axis of the cone.

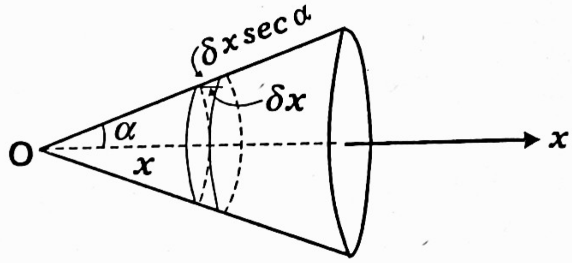

Fig. 4.14

Example 12. *A hollow right circular cone.*

Sol. The cone can be thought of as consisting of infinitely many circular bands parallel to the base. We take the symmetry axis as x-axis and the origin at the vertex of the cone (Fig. 4·14).

The mass of a circular band at distance x from O

$$= \rho\, 2\pi\, x \tan\alpha . \delta x \sec\alpha,$$

where ρ is the density of the cone.

The *c.m.* of the band lies on the x-axis.

Therefore, the distance of the *c.m.* of the cone from O is given by

$$\bar{x} = \frac{\int_0^h \rho .\, 2\pi\, x \tan\alpha .\ \sec\alpha .\ x\, dx}{\int_0^h \rho .\ 2\pi\, x \tan\alpha \sec\alpha .\ dx}$$

$$= \frac{\int_0^h x^2\, dx}{\int_0^h x\, dx} = \frac{2}{3} h.$$

Example 13. *A uniform solid hemisphere.*

Sol. Let a be the radius of the hemisphere, and O its centre. The line through O perpendicular to the plane face is an axis of symmetry and the *c.m.* of the solid, therefore, lies on this line which we take as the x-axis. We regard the solid as made up of infinitely many slices parallel to the plane face as in Fig. 4·15.

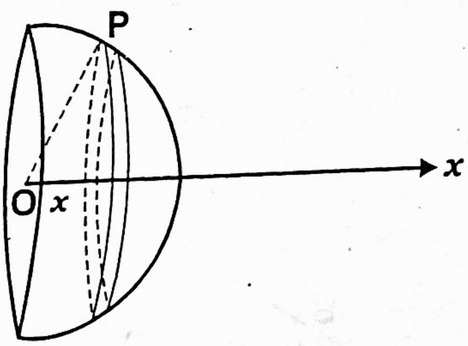

Fig. 4.15

Clearly the radius of the slice at distance x from O

$$= \sqrt{a^2 - x^2}.$$

$\therefore$ mass of the slice

$$= \rho . \pi\, (a^2 - x^2)\, \delta x,$$

where ρ is the density of the solid and δx is the thickness of the slice.

The *c.m.* of the slice lies on the *x*-axis, at distance *x* from O

$\therefore$ the position of the *c.m.* of the solid is given by

$$\bar{x}=\frac{\int_0^a \rho\,\pi\,(a^2-x^2)\,x\,dx}{\int_0^a \rho\,\pi\,(a^2-x^2)\,dx}=\frac{\left|\frac{a^2x^2}{2}-\frac{x^4}{4}\right|_0^a}{\left|a^2x-\frac{x^3}{3}\right|_0^a}$$

$$=\frac{\frac{a^4}{2}-\frac{a^4}{4}}{a^3-\frac{a^3}{3}}=\frac{3}{8}a.$$

Example 14. *A uniform circular arc.*

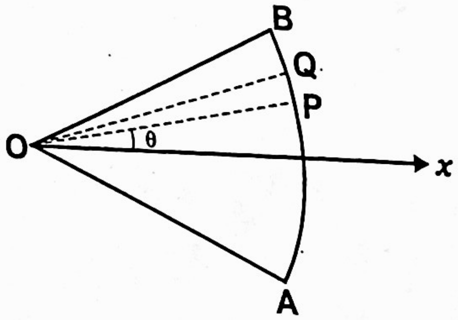

Fig. 4.16

Sol. Suppose the radius of the arc *AB* is *a* and its central angle is 2α (Fig. 4·16).

Let *Ox* be the bisector of the angle *AOB*. We take O as the origin and *Ox* as the *x*-axis.

Let $\widehat{PQ}$ be an element of the arc where measures of $\angle POX$, $\angle QOX$ are respectively θ and $\theta+\delta\theta$.

The mass of the element $\widehat{PQ}=\rho a\,\delta\theta$.

x-coordinate of the *c.m.* of the element

$=a\cos\theta$.

Therefore, the position of the *c.m.*, which by symmetry lies on *Ox*, is given by

$$\bar{x}=\frac{\int_{-\alpha}^{\alpha}\rho.a\cos\theta.a\,d\theta}{\int_{-\alpha}^{\alpha}\rho\,a\,d\theta}=\frac{\left|a\sin\theta\right|_{-\alpha}^{\alpha}}{\left|\theta\right|_{-\alpha}^{\alpha}}$$

$$=a\frac{\sin\alpha}{\alpha}.$$

Example 15. *A uniform sector of a circular lamina.*

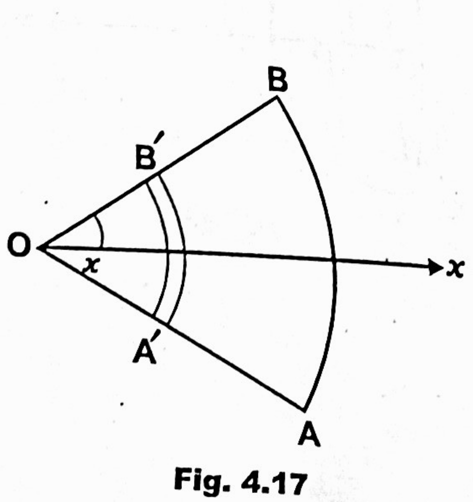

Fig. 4.17

Sol. Let *AOB* be a sector of a circular lamina of radius *a* (Fig. 4·17). Let the measure of $\angle AOB$ be 2α. By symmetry the *c.m.* lies on *Ox*, the bisector of $\angle AOB$. The sector may be regarded as consisting of strips concentric with the given circle as in Fig. 4·17.

The length of a strip $A'B'$ of radius x

$$=2\alpha.x.$$

$\therefore$ mass of this strip

$$=\rho\, 2\alpha.x\,\delta x,$$

where ρ is the density of the sector and δx is the breadth of the strip.

By the previous example the *c.m.* of the strip is at a distance $\frac{x\sin\alpha}{\alpha}$ from O.

$\therefore$ the position of the *c.m.* of the sector is given by

$$\bar{x}=\frac{\int_0^a \rho\, 2\alpha\, x\, dx.\frac{x\sin\alpha}{\alpha}}{\int_0^a \rho\, 2\alpha\, x\, dx}$$

$$=\frac{\sin\alpha}{\alpha}.\frac{\int_0^a x^2\, dx}{\int_0^a x\, dx}=\frac{2a}{3}.\frac{\sin\alpha}{\alpha}.$$

Example 16. Find the *c.m.* of a semi-circular lamina of radius a whose density varies as the square of the distance from the centre.

Sol. Let $\widehat{ABC}$ be the bounding semi-circle. By symmetry the *c.m.* lies on OB, the radius perpendicular to the diameter AOC as in Fig. 4·18. We may regard the lamina as consisting of infinitely many concentric semi-circular strips.

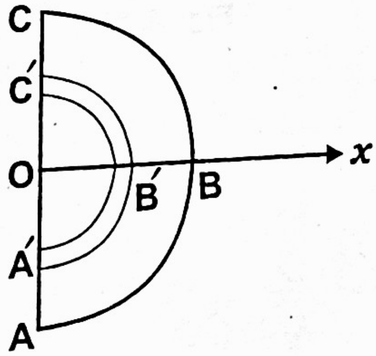

Fig. 4.18

The mass of a strip of radius x

$$=\pi\, x\, \lambda\delta x\, x^2,$$

where λ is a constant.

The distance of the *c.m.* of the strip from O

$$=\frac{2x}{\pi}.$$

(Replacing a by x and α by $\frac{\pi}{2}$ in the result of Ex. 14).

∴ the position of the *c.m.* of the whole lamina is given by

$$\bar{x} = \frac{\int_0^a \pi x \lambda\, x^2 \frac{2x}{\pi}\, dx}{\int_0^a \pi x \lambda\, x^2\, dx}$$

$$= \frac{2}{\pi} \frac{\int_0^a x^4\, dx}{\int_0^a x^3\, dx} = \frac{2}{\pi} \frac{\frac{a^5}{5}}{\frac{a^4}{4}}$$

$$= \frac{8a}{5\pi}.$$

4.7 Centre of Mass of an Arc

We have found the *c.m.* of arcs in some special cases from considerations of symmetry. We now derive general formulae for the coordinates of the *c.m.* of any arc of a given curve shown in Fig. 4·19.

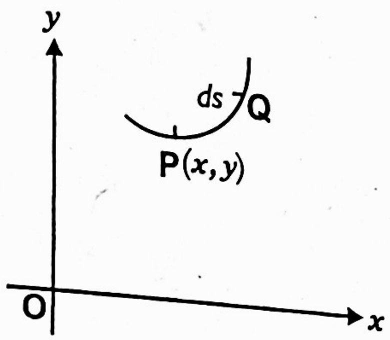

Fig. 4.19

Let ρ be the density of a plane curve at the point (x, y) and ds the arc element at the same point. Then an element of mass is given by

$$dm = \rho\, ds$$

∴ formulae for the *c.m.* become

$$\bar{x} = \frac{\int \rho x\, ds}{\int \rho\, ds}, \quad \bar{y} = \frac{\int \rho y\, ds}{\int \rho\, ds}. \tag{4.19}$$

In case of a uniform arc the *c.m.* (which now coincides with the centroid of the arc) is given by

$$\bar{x} = \frac{\int x\, ds}{\int ds}, \quad \bar{y} = \frac{\int y\, ds}{\int ds}. \tag{4.20}$$

N.B. In terms of Cartesian coordinates

$$ds^2 = dx^2 + dy^2 \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and in terms of polar coordinates

$$ds^2 = dr^2 + r^2\, d\theta^2 \quad \text{or} \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

If x, y are given in terms of a parameter t, we may use the result

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$$

Example. Find the centroid of the arc of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ lying in the first quadrant.

Sol. ***First Method*** :

The equation of the curve gives

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0.$$

$$\therefore\quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

$$\therefore\quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}\,dx$$

$$=\frac{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{x^{\frac{1}{3}}}\,dx$$

$$=\left(\frac{a}{x}\right)^{\frac{1}{3}}dx. \quad [\because\ (x,y) \text{ lies on the given curve}]$$

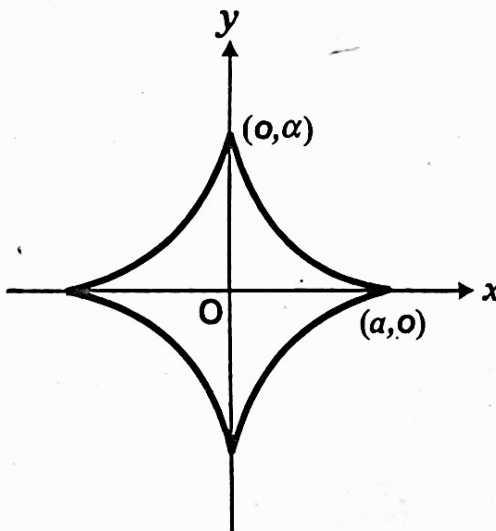

Fig. 4.20

$$\therefore\quad \bar{x}=\frac{\int_0^a x\left(\frac{a}{x}\right)^{\frac{1}{3}}dx}{\int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}}dx}=\frac{\int_0^a x^{\frac{2}{3}}\,dx}{\int_0^a x^{-\frac{1}{3}}\,dx}$$

$$=\frac{\left|\frac{3}{5}x^{\frac{5}{3}}\right|_0^a}{\left|\frac{3}{2}x^{\frac{2}{3}}\right|_0^a}=\frac{2}{5}a.$$

By symmetry $\bar{y}=\bar{x}=\frac{2}{5}a.$

Second Method :

The given curve has the parametric equations

$$x=a\cos^3\theta,\ y=a\sin^3\theta.$$

$$\therefore\ \frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta.$$

$$\therefore\ ds=\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$=3a\sin\theta\cos\theta\,d\theta.$$

Also, as x varies from 0 to a, θ varies from $\frac{\pi}{2}$ to 0.

$$\therefore\ \bar{x}=\frac{\int_{\pi/2}^{0} a\cos^3\theta .\, 3a\sin\theta\cos\theta\,d\theta}{\int_{\pi/2}^{0} 3a\sin\theta\cos\theta\,d\theta}$$

$$=a\frac{\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta}{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}=a\frac{\dfrac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\,\Gamma\left(\frac{4+1}{2}+1\right)}}{\dfrac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\,\Gamma\left(\frac{1+1}{2}+1\right)}}$$

$$=a\frac{\dfrac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(\frac{7}{2})}}{\dfrac{\Gamma(1)\Gamma(1)}{\Gamma(2)}}$$

$$=\frac{\dfrac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}}{\dfrac{1}{1}}\,a=\frac{2}{5}\,a=\bar{y}\text{ (By symmetry).}$$

4.8 Centroid of a Plane Region

(1) Consider the region bounded by the curve $y=f(x)$, the x-axis, and the ordinates $x=a$, $x=b$ as shown in Fig. 4·21. The area of an elementary strip parallel to y-axis at distance x from it is $y\,\delta x$.

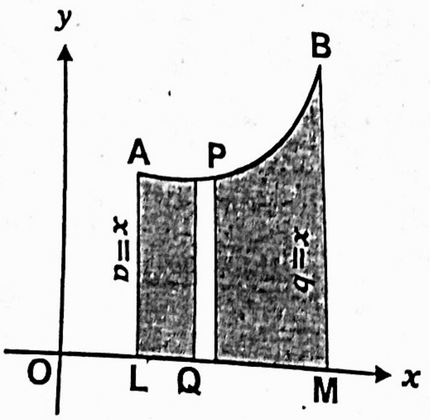

Fig. 4.21

The coordinates of the centroid of the strip are $\left(x, \frac{y}{2}\right)$.

$\therefore$ coordinates of the centroid of the given region are

$$\bar{x}=\frac{\int_a^b xy\,dx}{\int_a^b y\,dx},$$

$$\bar{y}=\frac{\int_a^b \frac{y}{2}\cdot y\,dx}{\int_a^b y\,dx}=\frac{1}{2}\frac{\int_a^b y^2\,dx}{\int_a^b y\,dx}.$$

(2) Formulae for the centroid of the region bounded by the curve $r=f(\theta)$ and the radius vectors $\theta=\theta_1$, $\theta=\theta_2$ may be obtained as follows :

Let $\widehat{AB}$ be the curve $r=f(\theta)$ and OA, $\bar{OB}$ the radius vectors $\theta=\theta_1$, $\theta=\theta_2$ as shown in Fig. 4·22. Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points of the curve.

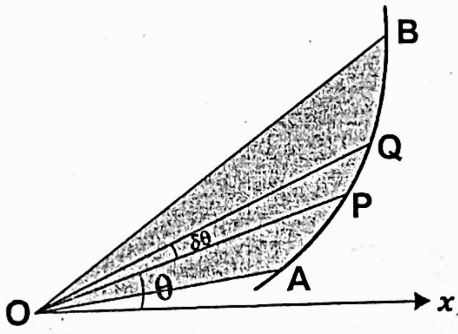

Fig. 4.22

Then the area of the sectorial element POQ

$=\frac{1}{2}r(r+\delta r)\sin\delta\theta$

$=\frac{1}{2}r(r+\delta r)\,\delta\theta,$

(Since $\delta\theta$ in the limit tends to zero.

$=\frac{1}{2}r^2\,\delta\theta.$

(To first order small quantities.)

Also the coordinates of the centroid of the elementary area are $(\frac{2}{3}r\cos\theta, \frac{2}{3}r\sin\theta)$. Hence the centroid of the total area is given by

$$\bar{x}=\frac{\int_{\theta_1}^{\theta_2}\frac{2}{3}r\cos\theta\cdot\frac{1}{2}r^2 d\theta}{\int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 d\theta}=\frac{\frac{2}{3}\int_{\theta_1}^{\theta_2} r^3\cos\theta\,d\theta}{\int_{\theta_1}^{\theta_2} r^2 d\theta},$$

$$\bar{y}=\frac{\int_{\theta_1}^{\theta_2}\frac{2}{3}r\sin\theta\cdot\frac{1}{2}r^2 d\theta}{\int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 d\theta}=\frac{\frac{2}{3}\int_{\theta_1}^{\theta_2} r^3\sin\theta\,d\theta}{\int_{\theta_1}^{\theta_2} r^2 d\theta}.$$

Example 1. Find the centroid of the region bounded by the coordinate axes and the circle $x^2+y^2=a^2$ which lies in the first quadrant.

Sol.
$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx}=\frac{\int_0^a x\sqrt{a^2-x^2}\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx}$$

$$=\frac{\int_0^a \frac{1}{2}\cdot 2x\sqrt{a^2-x^2}\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx}=\frac{\left|-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right|_0^a}{\left|\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a}$$

$$=\frac{\frac{a^3}{3}}{a^2\frac{\pi}{4}}=\frac{4a}{3\pi}=\bar{y}\quad \text{(By symmetry).}$$

Example 2. Find the distance from the cusp of the centroid of the region bounded by the cardioide

$$r=a(1+\cos\theta).$$

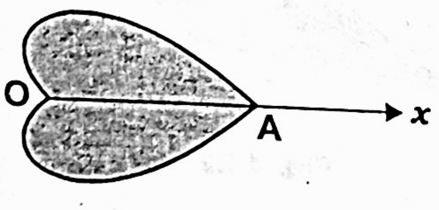

Fig. 4.23

Sol. By symmetry the centroid lies on the initial line. Also the x-coordinate of the centroid of the upper half is the same as that of the whole region (*see* Fig. 4·23).

$\therefore$ taking limits of θ for the upper half only,

$$\bar{x}=\frac{\frac{2}{3}\int_0^\pi r^3\cos\theta\,d\theta}{\int_0^\pi r^2\,d\theta}$$

$$=\frac{\frac{2}{3}\int_0^\pi a^3(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^\pi a^2(1+\cos\theta)^2\,d\theta}$$

$$=\frac{2}{3}a.\frac{\int_0^\pi(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta}{\int_0^\pi(1+2\cos\theta+\cos^2\theta)\,d\theta}$$

The integral in the numerator is

$$I_1=\int_0^{\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta.$$

Now $\cos(\pi-\theta)=-\cos\theta$ and $\cos^3(\pi-\theta)=-\cos^3\theta$.

But $\cos^2(\pi-\theta)=\cos^2\theta$, $\cos^4(\pi-\theta)=\cos^4\theta$.

$$\therefore\quad I_1=2\int_0^{\pi/2}(3\cos^2\theta+\cos^4\theta)\,d\theta.$$

$$=2\left(3.\frac{1}{2}\cdot\frac{\pi}{2}+\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)=2.\frac{15}{16}\pi$$

$$=\frac{15}{8}\pi.$$

Similarly, the integral in the denominator

$$I_2=\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=2\int_0^{\pi/2}(1+\cos^2\theta)\,d\theta$$

$$=2.\left(\frac{\pi}{2}+\frac{1}{2}\cdot\frac{\pi}{2}\right)=\frac{3}{2}\pi.$$

$$\therefore\quad \bar{x}=\frac{2}{3}a\times\frac{\frac{15}{8}\pi}{\frac{3}{2}\pi}=\frac{2}{3}\times\frac{15}{8}\times\frac{2}{3}a$$

$$=\frac{5}{6}a.$$

4.9 Mass Centre of a Solid of Revolution

To find the centre of mass of a solid of revolution, i.e., a solid whose boundary is obtained by revolving a given plane curve, about a line in its plane.

Let the given curve be

$$y=f(x)$$

and the given line be taken as the x-axis (Fig. 4·24). Consider the variation of x from $x=a$ to $x=b$.

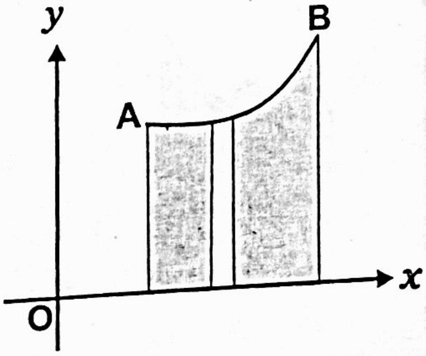

Fig. 4.24

The solid can be regarded as made up of thin circular slices perpendicular to the x-axis. The mass of the slice at a distance x from the origin is $\rho\pi y^2\delta x$. Since the *c.m.*

of each slice lies on the x-axis the *c.m.* of the whole solid also lies on the same line, *i.e.*,

$$\bar{y}=0.$$

Also
$$\bar{x}=\frac{\int_a^b x.\rho\,\pi y^2\,dx}{\int_a^b \rho\,\pi y^2\,dx}=\frac{\int_a^b x\,y^2\,dx}{\int_a^b y^2\,dx}\quad\text{(when } \rho \text{ is constant).}$$

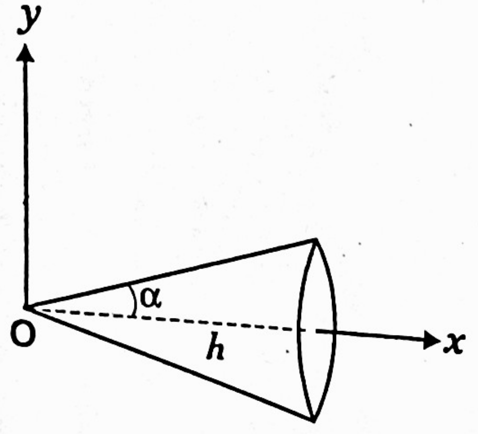

Fig. 4.25

Example. *A solid right circular cone.*

Sol. A solid right circular cone can be regarded as a solid of revolution formed by the rotation of the line

$$y=\tan\alpha . x, \qquad (0 \leqslant x \leqslant h)$$

about the x-axis (Fig. 4·25).

$$\therefore \quad \bar{x}=\frac{\int_0^h x y^2\,dx}{\int_0^h y^2\,dx}=\frac{\int_0^h x^3 \tan^2\alpha\,dx}{\int_0^h x^2\tan^2\alpha\,dx}$$

$$=\frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx}=\frac{\left|\frac{x^4}{4}\right|_0^h}{\left|\frac{x^3}{3}\right|_0^h}$$

$$=\frac{3}{4}h.$$

4.10 Mass Centre of a Surface of Revolution

To find the c.m. of a surface of revolution, i.e., surface formed by the rotation of a plane curve about a line in its plane.

Let the plane curve be $y=f(x)$ where x varies from $x=a$ to $x=b$, and let the rotation axis be taken as the x-axis (*see* Fig. 4·26). Then the area of an element of the surface of revolution is $2\pi y\,\delta s$.

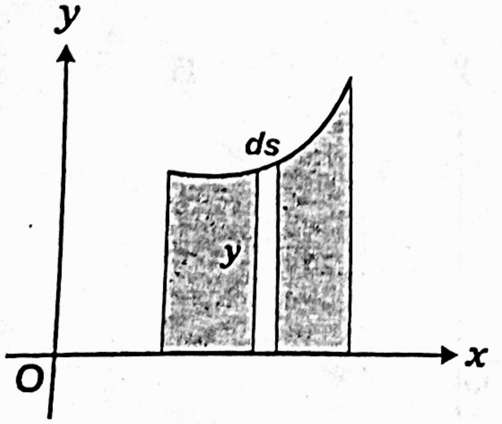

Fig. 4.26

Clearly $\bar{y}=0$.

Also $\bar{x}=\dfrac{\int \rho\, x\, 2\pi y\, ds}{\int \rho\, 2\pi y\, ds}$

$=\dfrac{\int \rho\, x y\, ds}{\int \rho\, y\, ds}$

$=\dfrac{\int x y\, ds}{\int y\, ds}$ (when the surface density is constant).

N.B. In a numerical case the value of ds in terms of Cartesian coordinates $\left(ds=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx\right)$, or polar coordinates $\left(ds=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\, d\theta\right)$, or a parameter $\left(ds=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\, dt\right)$ may be substituted as found convenient.

Example 1. Find the *c. m.* of a hollow right circular cone of semi-vertical angle α and height h.

Sol. A hollow right circular cone may be regarded as the surface of revolution generated by the rotation of the line

$$y=\tan\alpha . x \qquad (0 \leqslant x \leqslant h)$$

about the x-axis (Fig. 4·27).

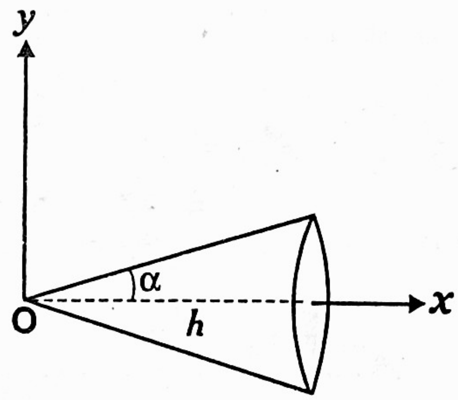

Fig. 4.27

By symmetry $\bar{y}=0$.

Also $\bar{x}=\dfrac{\int x y\, ds}{\int y\, ds}$

Now $\dfrac{dy}{dx}=\tan\alpha$.

$\therefore$ $ds=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx=\sqrt{1+\tan^2\alpha}\, dx$.

$=\sec\alpha . dx$

$\therefore$ $\bar{x}=\dfrac{\int_0^h x . x \tan\alpha . \sec\alpha\, dx}{\int_0^h x \tan\alpha . \sec\alpha\, dx}$

$=\dfrac{\int_0^h x^2\, dx}{\int_0^h x\, dx}=\dfrac{2}{3}h.$

Example 2. Find the *c.m.* of the surface generated by the revolution of the arc of the parabola, lying between the vertex and the latus rectum, about the x-axis.

Sol. Let the parabola be

$$y^2=4a\,x,$$

so that $y=2\sqrt{a}\sqrt{x}.$

$$\frac{dy}{dx}=2\sqrt{a}\cdot\frac{1}{2\sqrt{x}}=\sqrt{\frac{a}{x}}.$$

$$\therefore \quad ds=\sqrt{1+\frac{a}{x}}\,dx.$$

By symmetry $\bar{y}=0$.

$$\bar{x}=\frac{\int x\,y\,ds}{\int y\,ds}=\frac{\int_0^a x.\,2\sqrt{ax}.\sqrt{1+\frac{a}{x}}\,dx}{\int_0^a 2\sqrt{ax}\sqrt{1+\frac{a}{x}}\,dx}$$

$$=\frac{\int_0^a x\sqrt{a+x}\,dx}{\int_0^a \sqrt{a+x}\,dx}$$

$$=\frac{\int_0^a (x+a-a)\sqrt{x+a}\,dx}{\int_0^a \sqrt{x+a}\,dx}$$

$$=\frac{\int_0^a\left[(x+a)^{\frac{3}{2}}-a\sqrt{x+a}\right]dx}{\int_0^a \sqrt{x+a}\,dx}$$

$$= \frac{\left| \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2}{3}a(x+a)^{\frac{3}{2}} \right|_0^a}{\left| \frac{2}{3}(x+a)^{\frac{3}{2}} \right|_0^a}$$

$$= \frac{10 + 6\sqrt{2}}{35}$$

.11 Theorems of Pappus

In certain cases in which we know the area of a surface or the volume of a solid, we can easily compute the position of the mass-centre with the help of two theorems called Theorems of Pappus.

Theorem 1. *Let there be a uniform distribution of mass along a plane curve C of length s lying entirely on one side of a line l which lies in the plane of the curve, and let p be the distance of the mass-centre of C from l. If S is the surface area generated by the rotation of C about l, then*

$$2\pi p s = S,$$

i.e., $$p = \frac{S}{2\pi s}.$$

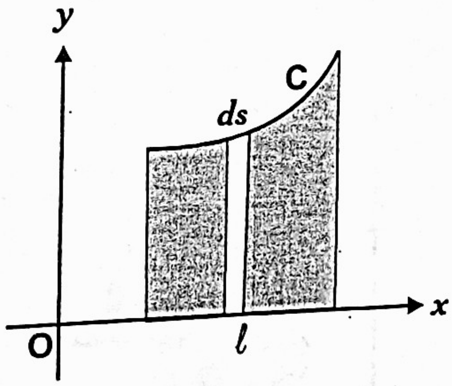

Fig. 4.28

Proof. We choose the coordinate system such that l is the x-axis (Fig. 4·28). In such a case,

$$p = \bar{y} \text{ for } C$$

$$= \frac{\int \rho y\, ds}{\int \rho\, ds} = \frac{\int y\, ds}{s}, \quad (\because \rho \text{ is constant})$$

$$\therefore \quad \int y\, ds = ps. \qquad \ldots (4{\cdot}20)$$

Also S = surface area generated by rotation of C about l (the x-axis)

$$= \int 2\pi y\, ds$$

$$= 2\pi \int y\, ds$$

$$= 2\pi p s, \qquad \text{(by 4·20)}$$

which proves the theorem.

Example. *A semi-circular wire.*

Sol. We know that the surface area of a sphere of radius a is

$$S = 4\pi a^2.$$

But the sphere can be regarded as the surface of revolution generated by the rotation of a semi-circle about the diameter joining its ends. The length of the semi-circle is

$$s = \pi a.$$

$\therefore$ distance of the centroid (or *c.m.* in case of uniform semi-circular wire) of the semi-circle from the diameter is

$$p = \frac{S}{2\pi s} = \frac{4\pi a^2}{2\pi . \pi a} = \frac{2a}{\pi}.$$

Theorem 2. *Let there be a uniform distribution of mass on a plane region R, lying entirely on one side of a line l in the plane of the region. Let p be the distance of the mass-centre of R from l, and A the area of R. If V is the volume generated by the rotation of R about l, then*

$$2\pi p A = V,$$

i.e.,
$$p = \frac{V}{2\pi A}.$$

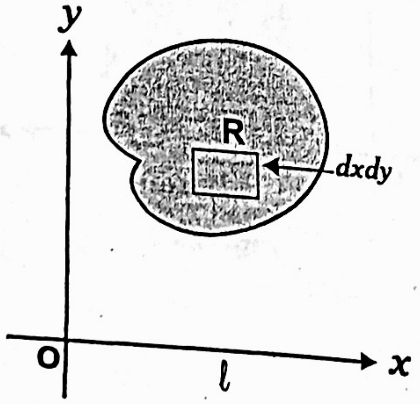

Fig. 4.29

Proof. We choose the coordinate system such that l is the x-axis and the plane of the region R is xy-plane (Fig. 4·29).

An area-element in R is $dx\, dy$

$$\therefore \quad p = \bar{y} \text{ for } R$$

$$= \frac{\int \rho\, y\, dx\, dy}{\int \rho\, dx\, dy} = \frac{\int y\, dx\, dy}{\int dx\, dy}, \qquad (\because \rho \text{ is constant})$$

$$= \frac{\int y\, dx\, dy}{A}, \qquad (\because \int dx\, dy = A)$$

$$\therefore \int y\, dx\, dy = A p. \qquad \ldots (4{\cdot}21)$$

Now
$$V = \int 2\pi y\, dx\, dy$$
$$= 2\pi \int y\, dx\, dy$$
$$= 2\pi p . A. \qquad \text{(by 4·21)}$$

This proves the theorem.

Example. *A semi-circular plane lamina.*

Sol. We know that the volume of a solid sphere of radius a is

$$V=\frac{4}{3}\pi a^3.$$

But the sphere can be regarded as a solid of revolution generated by the rotation of a semi-circular lamina about its bounding diameter. The area of the semi-circular lamina is

$$A=\tfrac{1}{2}\pi a^2.$$

Therefore, distance of the centroid (*c.m.*) of the lamina from the diameter is

$$p=\frac{V}{2\pi A}=\frac{\frac{4}{3}\pi a^3}{2\pi\,\frac{1}{2}\pi a^2}=\frac{4a}{3\pi}.$$

We have seen (*see* Art. 2·15) that a system of parallel forces either has a resultant or reduces to a couple. In the former case the resultant passes through a particular point (the centre of parallel forces) for all the orientations of the forces provided their magnitudes and points of application are kept unchanged.

In the following article we discuss an important case of parallel forces.

4.12 Centre of Gravity

The result proved in Art. 2·15 has an important application. In case of a body within the gravitational field of the earth, each element of matter in it is attracted towards the centre of the earth. The force of attraction exerted on an element is called its weight. If the size of the body is small as compared with that of the earth (as is the case in almost all practical cases), the forces on all the elements will be parallel (*see* Fig. 4·30) and will have a resultant, equal to the weight of the body. The resultant weight will, for all orientations of the body, pass through a particular point G of the body. This point is called the *centre of gravity* of the body.

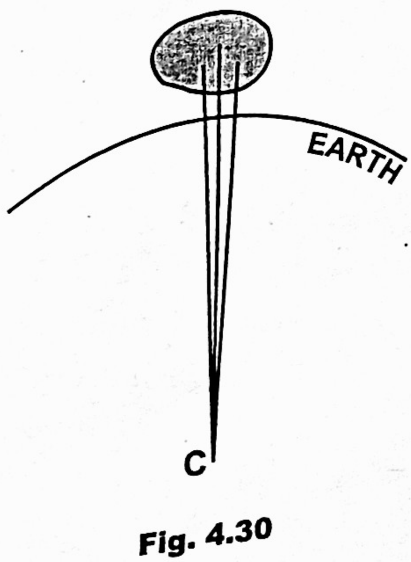

Fig. 4.30

The centre of gravity (*c.g.*) of a set of particles $m_1, \ldots, m_n$ with position vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ has clearly the position vector

$$\bar{\mathbf{r}} = \frac{\Sigma\, m_i\, g\, \mathbf{r}_i}{\Sigma\, m_i\, g}.$$

Since g is the same for all the particles, the last formula becomes

$$\bar{\mathbf{r}} = \frac{\Sigma\, m_i\, \mathbf{r}_i}{\Sigma\, m_i}. \qquad (4.22)$$

For a rigid body or any continuous distribution of matter, the corresponding formula is

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{\int dm}. \qquad (4.23)$$

Formulae (4·22) and (4·23) show that the *c.g.* of a set of particles or a rigid body (when it exists) coincides with its *c.m.* The Cartesian equivalents of equations (4·22) and (4·23) have already been written in the course of discussion of mass-centres.

Theorem. *The potential energy of a body is equal to that of a single particle with mass equal to the total mass of the body, situated at its centre of gravity.*

Proof. If (x, y, z) are the coordinates of an element dm of the body, and z-axis is vertical, the total potential energy V of the body is given by

$$V = \int g\, z\, dm$$
$$= g \int z\, dm$$
$$= g\, \frac{\int z\, dm}{\int dm} \int dm$$
$$= g\, \bar{z}\, m$$
$$= m\, g\, \bar{z},$$

where $m = \int dm$ is the total mass of the body and $\bar{z}$ is the z-coordinate of the *c.g.* This proves the theorem.

Exercises Set 4

1. A uniform rod AB is 4 ft. long and weighs 6 lb. and weights are attached to it as follows: 1 lb. at A, 2 lb. at 1 ft. from A, 3 lb. at 2 ft. from A, 4 lb. at 3 ft. from A and 5 lb. at B. Find the distance from A of the centre of gravity of the system. [**Ans.** 2·5 ft.]

2. Weights of 1, 2, 3, 4 lb. are placed at the corners A, B, C, D respectively of a square of side 8 inches. Find the distances of the *c.g.* of the set of weights from AB and AD. [Ans. 5·6 in., 4 in.]

3. Weights of 5, 1, 3, 2, 4 and 15 lb. are placed at the angular points of a regular hexagon taken in order. Find the distance of their *c.g.* from the 15 lb. weight.

[Ans. Half the length of a side on the line joining the 15 lb. and 3 lb. weights]

4. ABC is an isosceles triangular lamina in which $AB=AC=15$ inches, $BC=24$ inches. The weight of the lamina is 24 lb. and weights of 6, 6 and 4 lb. are placed at the corners A, B and C respectively. Find the distance of the *c.g.* of the system from BC.

[Ans. 3·37 in.]

5. The radius of the faces of a frustum of a solid cone are 2 ft. and 3 ft. and the height of the frustum is 4 ft. Find the distance of the *c.g.* from the larger face.

$$\left[\text{Ans. } \frac{33}{19} \text{ ft.}\right]$$

6. $ABCD$ is a trapezium which bounds a uniform lamina. AB, CD are parallel, and of lengths a, b respectively. Prove that the distance of the *c.m.* of the lamina from AB is

$$\frac{1}{3}\,h\,\frac{a+2b}{a+b},$$

where h is the distance between parallel sides.

7. A portion of a circular disc of radius r is cut off by a straight cut of length $2c$. Find the position of the *c.m.* of the larger portion.

If $r=1$ ft., $c=6$ in., calculate the distance of the mass-centre from the centre of the circle.

[Ans. 0·328 in.]

8. A square lamina $ABCD$ is divided into two parts by joining A to E, the middle point of BC. Prove that the line joining the *c.m.* of the triangular portion ABE to that of the quadrilateral portion $AECD$ is perpendicular to AE.

9. A lamina is in the shape of a square described on the base of an isosceles triangle. Find the tangent of the semi-vertical angle of the triangle if the *c.m.* of the whole lamina is at the middle point of the base.

[**Ans.** $\frac{1}{2\sqrt{3}}$]

10. From a uniform circular disc of radius a, a circular hole, having radius half that of the disc, is punched. Find the position of the *c.m.* of the remainder.

[**Ans.** At distance $\frac{1}{6}$ a from the centre of the disc.]

11. From a semi-circular lamina of radius $2a$ a circular lamina of radius a is removed. Prove that the *c.m.* of the remainder is at a distance $\frac{16a}{3\pi} - a$ from the diameter.

12. Two uniform solid spheres, composed of the same material and whose diameters are 6 in. and 12 in. respectively, are firmly united. Find the *c.m.* of the combined body.

[**Ans.** One inch from the centre of the larger sphere]

13. A rod of length $5a$ is bent so as to form 5 sides of a regular hexagon. Show that the distance of its *c.m.* from either end of the rod is $\frac{\sqrt{133}}{10}$ a.

14. Find the *c.g.* of a semi-circular lamina of radius r when the density varies as the cube of the distance from the centre.

[**Ans.** $\frac{5r}{3\pi}$ from the centre along radius of symmetry.]

15. The density at any point in a sector of a circular lamina varies as the distance from the centre. Find the *c.g.* of the sector.

[**Ans.** $\frac{3a \sin\alpha}{4\alpha}$]

16. An isosceles triangular lamina is such that its mass per unit area at every point is proportional to the sum of the distances of the point from the equal sides of the triangle. Prove that the distance of the *c.m.* from the vertex is three-fourths of the altitude.

17. Find the centroid of the arc of the cycloid

$$x = a\,(\theta + \sin\theta),$$
$$y = a\,(1 - \cos\theta),$$

which lies in the first quadrant.

[**Ans.** $\left(\frac{1}{3}\,a(3\pi - 4), \frac{2a}{3}\right)$]

18. Find the position of the centroid of a quadrant of an elliptic lamina.

[**Ans.** $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$]

19. A lamina is bounded by the astroid

$$x = a\cos^3\theta,\ y = a\sin^3\theta.$$

Find the centroid of its portion that lies in the first quadrant.

[**Ans.** $\left(\frac{256\,a}{315\,\pi}, \frac{256\,a}{315\,\pi}\right)$]

20. Find the centroid of the surface formed by the revolution of the cardioide $r = a\,(1 + \cos\theta)$ about the initial line.

[**Ans.** $\left(\frac{50}{63}\,a, 0\right)$]

21. Show that the *c.m.* of a segment of a solid sphere of radius a, at a distance b from the centre of the sphere is at a distance

$$\frac{3}{4}\,\frac{(a+b)^2}{2a+b}$$

from the centre.

22. Prove that the *c.m.* of a hemispherical shell of radius a is at a distance $\frac{a}{2}$ from the centre.

23. Show that the *c.g.* of the lamina bounded by a loop of the lemniscate $r^2 = a^2\cos 2\theta$ is on the initial line at a distance $\frac{\pi a}{4\sqrt{2}}$ from the pole.

24. Find the position of the *c.g.* of an octant of a uniform solid sphere.

[**Ans.** At a distance $\frac{3a}{8}$ from each plane face]

25. If the *c.g.* of a quadrilateral lamina is the same as that of four equal particles placed at its angular points, show that the bounding quadrilateral must be a parallelogram.