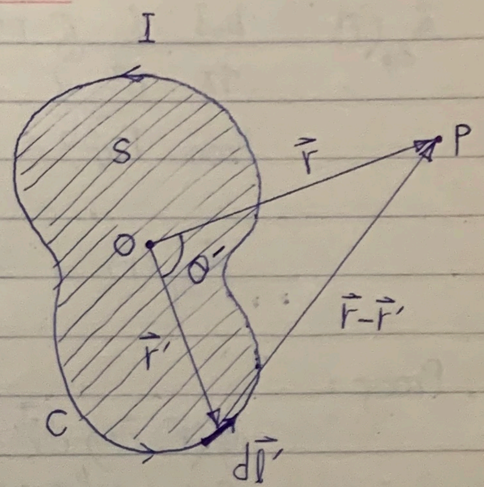
Multipole Expansion of the Vector Potential:



The magnetic vector potential at point 'P' due to current carrying circuit is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

C → entire length of the current loop.

Proved: $$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{r'^{l}}{r^{l+1}} P_l(\cos\theta')$$

$$\therefore \quad \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \oint_C r'^{l} P_l(\cos\theta')\, d\vec{l}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint_C d\vec{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint_C r'\cos\theta'\, d\vec{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint_C r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l}' + \cdots$$

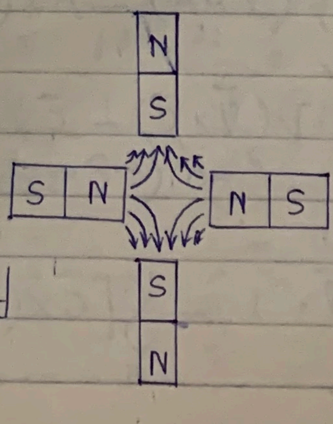
$$\vec{A}(\vec{r}) = \underset{\substack{\downarrow \\ l=0}}{\vec{A}_{mono}(\vec{r})} + \underset{\substack{\downarrow \\ l=1}}{\vec{A}_{dip}(\vec{r})} + \underset{\substack{\downarrow \\ l=2}}{\vec{A}_{quad}(\vec{r})} + \cdots$$

$$\vec{A}_{mono}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint_C d\vec{l}' = \text{Magnetic monopole term } \left(\sim \frac{1}{r}\right)$$

$= 0$ consequence of Maxwell's equation $\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$

$\oint_C d\vec{l}' =$ total vector displacement around a closed loop $= 0$!

There are no (N/S) magnetic charges / no magnetic monopoles have been (conclusively/convincingly) ever observed in our universe.



Magnetic quadrupole

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint_C r'\cos\theta' \, d\vec{l}' \equiv \text{Magnetic dipole term } \left(\sim \frac{1}{r^2}\right)$$

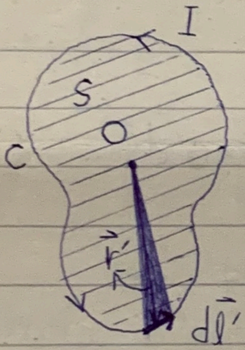
From Fig., $r'\cos\theta' = \hat{r}\cdot\vec{r}' = \dfrac{\vec{r}\cdot\vec{r}'}{r}$

$$\therefore \; \vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint_C (\vec{r}\cdot\vec{r}')\, d\vec{l}'$$

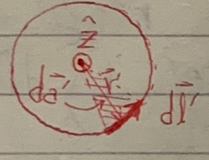
Proof :

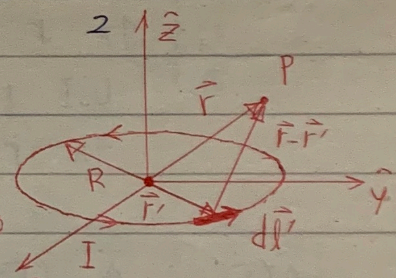
$$\oint_C (\vec{r}\cdot\vec{r}')\, d\vec{l}' = -\vec{r}\times\int_S d\vec{a}'$$

where, $\int_S d\vec{a}' \equiv \vec{a}' = \frac{1}{2}\oint_C \vec{r}'\times d\vec{l}' \equiv$ vector area of the contour loop / area enclosed by the current loop.



Area of the triangle $= d\vec{a}' = \frac{1}{2}\vec{r}'\times d\vec{l}'$





$$\vec{a}' = \frac{1}{2}\oint_C \vec{r}'\times d\vec{l}' = \frac{1}{2}\oint_C R\hat{r}\times R\,d\phi\,\hat{\phi} = \frac{1}{2}R^2\hat{z}\int_0^{2\pi} d\phi = \pi R^2\hat{z} \rightarrow \text{It works} ☺$$

Stoke's theorem : $\oint_C \vec{V}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a}$

consider a special case with $\vec{V} = \vec{c}\,T$;
where $\vec{c}$ is a constant vector and T is a scalar function.

$$\vec{c} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$$

$$\oint_C \vec{c}T\cdot d\vec{l} = \int_S [\vec{\nabla}\times(\vec{c}T)]\cdot d\vec{a}$$

Using, $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$

$$\therefore \; \vec{\nabla}\times(\vec{c}T) = T\underbrace{(\vec{\nabla}\times\vec{c})}_{=0\ (\because\ \vec{c}\text{ is constant vector})} - \vec{c}\times(\vec{\nabla}T)$$

$$\therefore \; \oint_C \vec{c}T\cdot d\vec{l} = -\int_S [\vec{c}\times(\vec{\nabla}T)]\cdot d\vec{a}$$

$$\oint_C \vec{c}T \cdot d\vec{l} = -\int_S [\vec{c}\cdot(\vec{\nabla}T)] \times d\vec{a}$$

$$\oint_C \vec{c}\cdot T d\vec{l} = -\int_S [\vec{c}\cdot(\vec{\nabla}T)] \times d\vec{a}$$

$$\vec{c}\cdot \oint_C T d\vec{l} = -\vec{c}\cdot \int_S \vec{\nabla}T \times d\vec{a}$$

$$\Rightarrow \boxed{\oint_C T d\vec{l} = -\int_S \vec{\nabla}T \times d\vec{a}}$$

In our case,

Put $\vec{r} = \vec{c}$ (i.e., Position vector of field point 'P', i.e., $\vec{r}$ is a constant vector) $\Rightarrow$ If $\vec{r}$ isn't constant then it will decrease and we no more have dipole term; rather a quadrupole term.

$$\therefore \oint_C (\vec{c}\cdot\vec{r}\,')\, d\vec{l}\,' = -\vec{c}\times\int_S d\vec{a}\,'$$

Let, $T = \vec{c}\cdot\vec{r}\,'$

$$\therefore \oint_C T d\vec{l}\,' = -\int_S \vec{\nabla}' T \times d\vec{a}\,'$$

$$\therefore \vec{\nabla}' T = \vec{\nabla}'(\vec{c}\cdot\vec{r}\,') = \vec{c}\times\underbrace{(\vec{\nabla}'\times\vec{r}\,')}_{=0} + \vec{r}\,'\times\cancel{(\vec{\nabla}'\times\vec{c})}^{=0} + (\vec{c}\cdot\vec{\nabla}')\vec{r}\,' + \cancel{(\vec{r}\,'\cdot\vec{\nabla}')\vec{c}}^{=0}$$

$$= (\vec{c}\cdot\vec{\nabla}')\,\vec{r}\,'$$

$$= \left(c_x\frac{\partial}{\partial x'} + c_y\frac{\partial}{\partial y'} + c_z\frac{\partial}{\partial z'}\right)(x'\hat{x} + y'\hat{y} + z'\hat{z})$$

$$= c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$$

$$\therefore \oint_C (\vec{r}\cdot\vec{r}\,')\, d\vec{l}\,' = -\int_S \vec{r}\times d\vec{a}\,' = -\vec{r}\times\int_S d\vec{a}\,'$$