# Chapter 4

## CENTRES OF MASS AND GRAVITY

### Centre of mass of a set of particles

**Def.**
Consider a set of $n$ particles of masses $m_1, m_2, \dots, m_n$, situated at the points $P_1, P_2, \dots, P_n$ whose position vectors relative to an origin $O$ are $\vec{r}_1, \vec{r}_2, \dots \vec{r}_n$. Then the linear moment of the set of particles with respect to $O$ is the vector

$$\sum_{i=1}^{n} m_i \vec{r}_i$$

**Def.** The centre of mass (C.m.) of a set of particles is the point with respect to which the linear moment of set is zero.

**Theorem** Every set of particles has one and only one centre of mass

**Proof** Let the particles $m_1, m_2, \dots m_n$ be located at the points $P_1(\vec{r}_1), P_2(\vec{r}_2), \dots P_n(\vec{r}_n)$

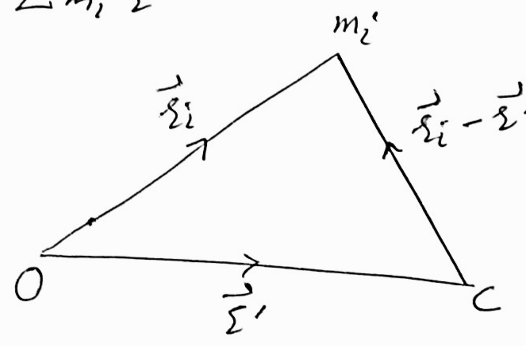
Suppose $C(\vec{r})$ is a c.m

p.v. of $P_i(\vec{r}_i)$ relative to $C$ is $\vec{r}_i - \vec{r}$

Therefore $0 = \sum m_i (\vec{r}_i - \vec{r}')$

$$= \sum m_i \vec{r}_i - \sum m_i \vec{r}'$$

$$\vec{r}' = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$




Suppose $C'(\vec{r}'')$ be another c.m. of set of particles.
Then linear moment of the given set of particles w.r.t $C'$ is zero if and only if

$$\vec{r}'' = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

Therefore

$$\vec{r}'' = \vec{r}'$$

i.e $C' \equiv C$

So c.m is unique.

In Cartesian coordinates

$$\vec{r}_i = (x_i, y_i, z_i)$$

and $\vec{r}\,' = (x', y', z')$

Then $x' = \dfrac{\sum m_i x_i}{\sum m_i}$, $y' = \dfrac{\sum m_i y_i}{\sum m_i}$,

$$z' = \frac{\sum m_i z_i}{\sum m_i}$$

## Centroid

The point

$$\frac{\sum \vec{r}_i}{n} = \left( \frac{\sum x_i}{n}, \frac{\sum y_i}{n}, \frac{\sum z_i}{n} \right)$$

is called the centroid of the points $P_1, P_2, \dots, P_n$.

In case when $m_1 = m_2 = \dots = m_n$, then c.m. of the particles is the same as centroid.

* In case of only two particles $m_1, m_2$

$$\vec{r}\,' = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \longrightarrow (1)$$

Note Eq. (1) gives the p.v. of the point dividing the directed line segment from $\vec{r}_1$ to $\vec{r}_2$ in the ratio

$m_2 : m_1$.

Therefore c.m of two particles $m_1, m_2$ divides the directed line segment joining them in the ratio $m_2 : m_1$.

In case $m_1 = m_2 = m$, then

$$\vec{r}' = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

which is centroid of the points $m_1, m_2$.

<u>Example 1</u> Find the centroid of the points $\hat{i}$, $2\hat{i} - \hat{j}$ and $3\hat{i} + \hat{j} - 4\hat{k}$. If particles of mass 2, 4, 3 grams are placed respectively at these points, what will be their c.m.?

<u>Solution</u>

The p.v. of the centroid

$$= \frac{\hat{i} + 2\hat{i} - \hat{j} + 3\hat{i} + \hat{j} - 4\hat{k}}{3}$$

$$= \frac{6\hat{i} - 4\hat{k}}{3}$$

The p.v. of c.m is

$$\vec{r}' = \frac{2(\hat{i}) + 4(2\hat{i} - \hat{j}) + 3(3\hat{i} + \hat{j} - 4\hat{k})}{2+4+3}$$

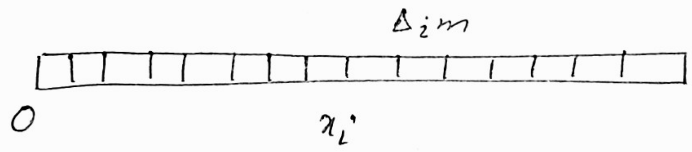
$$= \frac{19\hat{i} - \hat{j} - 12\hat{k}}{9}$$

## CENTRE OF MASS OF A CONTINUOUS DISTRIBUTION OF MATTER

### CENTRE OF MASS OF A THIN ROD

THIN ROD: A R.B. having length only and whose breath and thickness are negligible.

∴ Consider a thin rod of length $L$ and mass $m$




Subdivide the rod into $n$ parts. Let $\Delta_i m$ be the mass of the $i$th part. and $x_i$ be the distance from one end O of any point of $i$th part.

Then position of c.m is

$$x' = \lim_{\substack{n \to \infty \\ \max \Delta m \to 0}} \frac{\sum \Delta_i m \, x_i}{\sum \Delta_i m}$$

$$= \frac{\int_0^l x \, dm}{\int_0^l dm} =$$

$$= \frac{\int_0^l x \rho \, dx}{\int_0^l \rho \, dx}$$

$\rho = \frac{m}{V}$

$\rho = \frac{dm}{dx}$

In case of uniform rod (homogeneous rod)

$$x' = \int_0^l x\,dx \Big/ \int_0^l dx \longrightarrow (1)$$

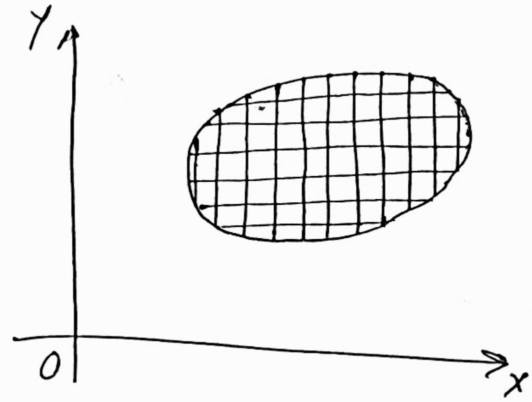
$$= \frac{x^2}{2}\Big|_0^l \Big/ x\Big|_0^l = \frac{l^2}{2} \Big/ l = \frac{l}{2}$$

$$x' = l/2$$

Eq. (1) defines the centroid of the rod. So in case of uniform rod, its c.m is the same as its centroid.

CENTRE OF MASS OF A LAMINA → Flat R-B with negligible thickness

Consider a lamina of mass $m$ and area $A$. Subdivide the lamina into rectangles by drawing lines parallel to coordinate axes.




Let $\Delta_i m$ be the mass of the $i$th rectangle whose diagonal is of length $\Delta_i l$ and let $\vec{r}_i = (x_i, y_i)$ be any point in it.

Then c.m of lamina is given by

$$\vec{r}' = \lim_{\substack{n \to \infty \\ \max \Delta_i l \to 0}} \frac{\sum_{i=1}^{n} \vec{r}_i \, \Delta_i m}{\sum_{i=1}^{n} \Delta_i m}$$

$$= \frac{\int \vec{r} \, dm}{\int dm}$$

$$= \frac{\int \vec{r} \rho \, dA}{\int \rho \, dA} \qquad \left| \begin{array}{l} m = \rho V \\ \because \; dm = \rho \, dA \end{array} \right.$$

where $\rho$ is density of the lamina.

I case of homogeneous lamina, the c.m is same as centroid.

$$\vec{r}' = \frac{\int \vec{r} \, dA}{\int dA} = \frac{\int \vec{r} \, dA}{A}$$

In terms of Cartesian coordinates

$\vec{r} = (x, y)$, $dA = dx\,dy$

Therefore above equation takes the form

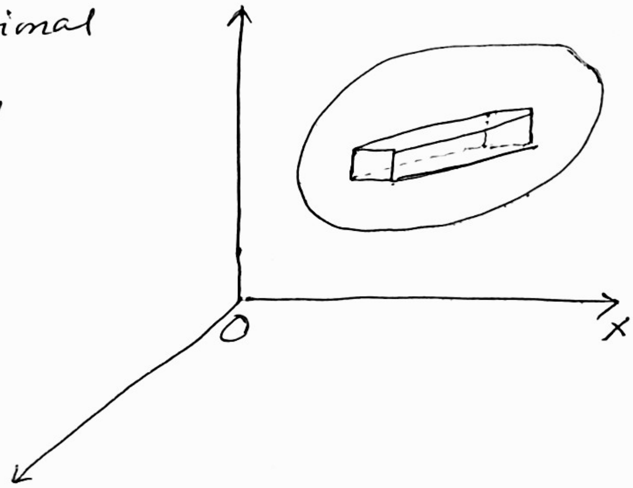
$$m x' = \iint \rho \, x \, dx\,dy$$

$$m y' = \iint \rho \, y \, dx\,dy$$

## CENTRE OF MASS OF A SOLID

Consider a three-dimensional R.B of mass $m$ and volume $V$.

Subdivide the R.B into rectangular parallelopiped by drawing planes parallel to the coordinate planes.




Let $\Delta_i m$ be mass of the $i$th parallelopiped whose diagonal is of length $\Delta_i l$ and $\vec{r}_i = (x_i, y_i, z_i)$ be any point within it.

Then

$$\vec{r}\,' = \lim_{\substack{n \to \infty \\ \max \Delta_i l \to 0}} \frac{\sum_{i=1}^{n} r_i \Delta_i m}{\sum_{i=1}^{n} \Delta_i m}$$

$$\vec{r}\,' = \frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \rho \vec{r}\, dv}{\int \rho\, dv} = \frac{\int \vec{r}\, dv}{\int dv} \quad \left(\text{If R.B is uniform}\right)$$

In terms of cartesian coordinates

$$m x' = \iiint \rho x \, dx\, dy\, dz, \quad m y' = \iiint \rho y \, dx\, dy\, dz$$

$$m z' = \iiint \rho x \, dx\, dy\, dz$$