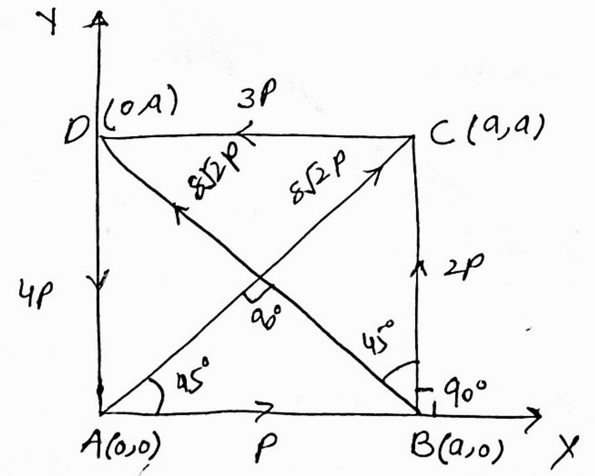
2.17 Examples

Example 1 Forces of magnitude $P, 2P, 3P, 4P$ act respectively along the sides AB, BC, CD, DA of a square ABCD, of side a, and forces each of magnitude $(8\sqrt{2})P$ act along the diagonals BD, AC. Find the magnitude of the resultant force and the distance of its line of action from A.

Solution

Consider the forces of magnitude P, $2P$, $3P$, $4P$ acting respectively along the sides AB, BC, CD, DA of a square ABCD each of side and forces each of magnitude $8\sqrt{2}P$ acting along the diagonals BD and AC.



Resolving the forces along the coordinate axes

$$X = P\cos 0° + 2P\cos 90° - 3P\cos 0° - 4P\cos 90°$$
$$+ 8\sqrt{2}P\cos 135° + 8\sqrt{2}P\cos 45°$$

$$X = P + 0 - 3P - 4P(0) + 8\sqrt{2}P\left(-\frac{1}{\sqrt{2}}\right) + 8\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{X = -2P}$$

$$Y = P\sin 0^\circ + 2P\sin 90^\circ - 4P\sin 0^\circ - 4P\sin 90^\circ + 8\sqrt{2}\sin 135^\circ + 8\sqrt{2}\sin 45^\circ$$

$$= 0 + 2P - 0 - 4P + 8\sqrt{2}P\left(\frac{1}{\sqrt{2}}\right) + 8\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$= -2P + 8P + 8P$$

$$\boxed{Y = 14P}$$

Let R be magnitude of the resultant force, then

$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{(-2P)^2 + (14P)^2}$$

$$= \sqrt{4P^2 + 196P^2}$$

$$= \sqrt{200}P$$

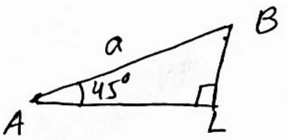
$$= 10\sqrt{2}P$$

Now Taking moments of forces about point A. Let G be the moment of the resultant force R, then by Varignon's Theorem

$$G = 2P(a) + 3P(a) + 8\sqrt{2}P\frac{a}{\sqrt{2}}$$

$$= 5Pa + 8Pa$$

$$= 13Pa$$



Now equation of line of action of resultant is

$$G - xY + yX = 0$$

$$13Pa - 14Px - 2Py = 0$$

or $\quad 13a - 14x - 2y = 0$

Let d be the distance of line of action of the resultant force from point A, Then

$$d = \frac{|13a - 14x - 2y|}{\sqrt{(-14)^2 + (-2)^2}}$$

Since At A, $x = 0, y = 0$

$$d = \frac{|13a - 0 - 0|}{\sqrt{196+4}} = \frac{13a}{\sqrt{200}}$$

$$\boxed{d = \frac{13a}{10\sqrt{2}}}$$

Example 3 Prove that any system of forces in a plane is equivalent to three suitable chosen forces X, Y, Z acting along the sides BC, CA, AB of a given triangle in the plane. Prove that if the system is equivalent to a couple G, Then

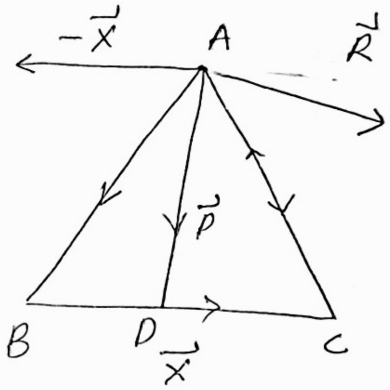
$$\frac{X}{BC} = \frac{Y}{CA} = \frac{G}{2\Delta}$$

Where Δ is the area of the triangle ABC.

Solution

Let the system of forces is reduced into a single force $\vec{R}$ and a single couple at point A.

We replace the couple by a force $\vec{X}$ through $\vec{BC}$ and an equal and opposite force $-\vec{X}$ at point A.



Let h_A be perpendicular from point A to BC, then

$$G_A = X \cdot h_A$$

The concurrent forces $\vec{R}$ and $-\vec{X}$ are equal to a single force $\vec{P}$ at A. Let the line of action of $\vec{P}$ cuts BC at point D, we can choose λ and μ so that

$$\vec{P} = (\lambda+\mu)\vec{AD}$$

then by (λ, μ) theorem

$$(\lambda+\mu)\vec{AD} = \lambda\vec{AB} + \mu\vec{AC}$$

$$\text{s.t} \quad BD : DC = \mu : \lambda$$

Now the system of forces is reduced to three forces $\vec{X}$ (through BC), $\vec{Y} = \mu\vec{CA}$ and $\vec{Z} = \lambda\vec{AB}$

Since the system is equivalent to a couple of moment G, so the sum of the moments of the forces about any point (A, B or C) will be G.

Taking the moment of forces about A

$$G = \vec{X} h_A + \vec{Y}(0) + \vec{Z}(0)$$

$$G = X h_A$$

Taking the moment of forces about B

$$G = X(0) + Y h_B + Z(0)$$

$$G = Y h_B$$

Similarly taking moment of forces about C

$$G = X(0) + Y(0) + Z h_z$$

$$G = Z h_z$$

So

$$X h_A = Y h_B = Z h_z = G$$

Now area of triangle ABC is given by

$$\Delta = \frac{1}{2} BC h_A \qquad \Delta = \frac{1}{2} CA h_B \qquad \Delta = \frac{1}{2} AB h_c$$

$$h_A = \frac{2\Delta}{BC} \qquad h_B = \frac{2\Delta}{CA}, \qquad h_c = \frac{2\Delta}{AB}$$

Again $X h_A = Y h_B = Z h_c = G$

then $X \frac{2\Delta}{BC} = Y \frac{2\Delta}{CA} = Z \cdot \frac{2\Delta}{AB} = G$

or $\boxed{\frac{X}{BC} = \frac{Y}{CA} = \frac{Z}{AB} = \frac{G}{2\Delta}}$

Exercise Set 2

Q1 If two forces P and Q act at such an angle that their resultant $R = P$ show that if P is doubled, the new resultant is at right angle to Q.

SOLUTION

Consider two forces P and Q inclined at an angle α. Let R be the magnitude of their resultant force, then

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

Since $R = P$

$$P = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$P^2 = P^2 + Q^2 + 2PQ\cos\alpha \quad \text{(Squaring both sides)}$$

$$0 = Q^2 + 2PQ\cos\alpha$$

or $\quad Q(Q + 2P\cos\alpha) = 0$

$\because Q \neq 0 \quad \therefore Q + 2P\cos\alpha = 0 \longrightarrow (1)$

If P is doubled and the new resultant makes an angle θ with Q, then

$$\tan\theta = \frac{2P\sin\alpha}{Q + 2P\cos\alpha}$$

$$= \frac{2P\sin\alpha}{0} \quad \text{(By (1))}$$

$$= \infty$$

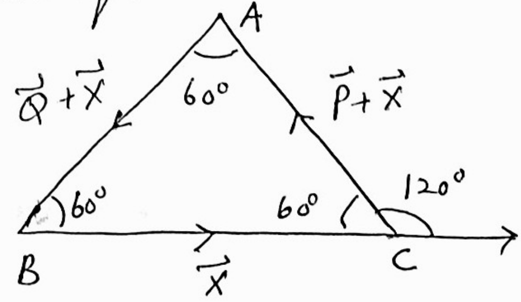
$$\theta = 90^\circ$$

The new resultant is at right angles to Q.

Q3 Forces $\vec{X}$, $\vec{P}+\vec{X}$, $\vec{Q}+\vec{X}$ act at a point in the directions of the sides of an equilateral triangle, taken one way round. Show that they are equivalent to two forces P and Q acting at an angle of 120°.

SOLUTION

Consider the forces $\vec{X}$, $\vec{P}+\vec{X}$, $\vec{Q}+\vec{X}$ acting at a point O parallel to the sides BC, CA, AB respectively of an equilateral triangle ABC. Resolving the forces along the coordinate axes



Let X' and Y' are sums of resolved parts of forces along Ox and Oy, then

$$X' = X\cos 0^\circ + (P+X)\cos 120^\circ + (Q+X)\cos 240^\circ$$

$$= X(1) + (P+X)(-\tfrac{1}{2}) + (Q+X)(-\tfrac{1}{2})$$

$$= X - \tfrac{1}{2}P - \tfrac{1}{2}X - \tfrac{1}{2}Q - \tfrac{1}{2}X$$

$$= X - X - \tfrac{1}{2}(P+Q)$$

$$= -\tfrac{1}{2}(P+Q)$$

$$Y' = X\sin 0^\circ + (P+X)\sin 120^\circ + (Q+X)\sin 240^\circ$$

$$= X(0) + (P+X)\left(\tfrac{\sqrt{3}}{2}\right) + (Q+X)\left(-\tfrac{\sqrt{3}}{2}\right)$$

$$= \tfrac{\sqrt{3}}{2}(P+X-Q-X)$$

$$y' = \frac{\sqrt{3}}{2}(P-Q)$$

Let R be the magnitude of the resultant force, Then

$$R = \sqrt{x'^2 + y'^2}$$

$$= \sqrt{\left[-\frac{1}{2}(P+Q)\right]^2 + \left(\frac{\sqrt{3}}{2}(P-Q)\right)^2}$$

$$= \sqrt{\frac{1}{4}(P^2+Q^2+2PQ) + \frac{3}{4}(P^2+Q^2-2PQ}$$

$$= \sqrt{\frac{1}{4}(P^2+Q^2+2PQ+3P^2+3Q^2-6PQ}$$

$$= \sqrt{\frac{1}{4}(4P^2+4Q^2-4PQ}$$

$$= \sqrt{\frac{1}{4}\cdot 4(P^2+Q^2-PQ)}$$

$$= \sqrt{P^2+Q^2+2PQ(-1/2)}$$

$$= \sqrt{P^2+Q^2+2PQ\cos(120^\circ)}$$

Hence system of forces is equivalent to two forces P and Q acting at an angle 120°.